Asymptotics for a special solution of the thirty fourth Painlevé equation

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November 24, 2008

Abstract

In a previous paper we studied the double scaling limit of unitary random matrix ensembles of the form $Z_{n,N}^{-1}|\det M|^{2\alpha}e^{-N\operatorname{Tr}V(M)}dM$ with $\alpha>-1/2$. The factor $|\det M|^{2\alpha}$ induces critical eigenvalue behavior near the origin. Under the assumption that the limiting mean eigenvalue density associated with V is regular, and that the origin is a right endpoint of its support, we computed the limiting eigenvalue correlation kernel in the double scaling limit as $n,N\to\infty$ such that $n^{2/3}(n/N-1)=O(1)$ by using the Deift-Zhou steepest descent method for the Riemann-Hilbert problem for polynomials on the line orthogonal with respect to the weight $|x|^{2\alpha}e^{-NV(x)}$. Our main attention was on the construction of a local parametrix near the origin by means of the ψ-functions associated with a distinguished solution u_α of the Painlevé XXXIV equation. This solution is related to a particular solution of the Painlevé II equation, which however is different from the usual Hastings-McLeod solution. In this paper we compute the asymptotic behavior of $u_\alpha(s)$ as $s\to\pm\infty$. We conjecture that this asymptotics characterizes u_α and we present supporting arguments based on the asymptotic analysis of a one-parameter family of solutions of the Painlevé XXXIV equation which includes u_α. We identify this family as the family of *tronquée* solutions of the thirty fourth Painlevé equation.

1 Introduction and statement of results

For $n\in\mathbb{N}$, $N>0$, and $\alpha>-1/2$, consider the unitary random matrix ensemble

$$Z_{n,N}^{-1}|\det M|^{2\alpha}e^{-N\operatorname{Tr}V(M)}\,dM, \tag{1.1}$$

on the space $\mathcal{M}(n)$ of $n\times n$ Hermitian matrices M, where V is real analytic and satisfies

$$\lim_{x\to\pm\infty}\frac{V(x)}{\log(x^2+1)}=+\infty. \tag{1.2}$$

Suppose also that the equilibrium measure μ_V for V is regular [6], and that 0 is a right endpoint of the support of μ_V.

In the paper [15] we computed the limiting eigenvalue correlation kernel in the double scaling limit as $n,N\to\infty$ such that $n^{2/3}(n/N-1)=O(1)$. We showed that it is characterized through a solution of a model RH problem associated with a special solution of the equation number XXXIV from the list of Painlevé and Gambier [11],

$$u''=4u^2+2su+\frac{(u')^2-(2\alpha)^2}{2u}. \tag{1.3}$$

In particular, we showed how the relevant solution $u(s)$ of (1.3) (which we denote by $u_\alpha(s)$) can be obtained from a solution of a model RH problem, which we next describe.

1.1 The model RH problem

The model RH problem is posed on a contour Σ in an auxiliary z-plane, consisting of four rays $\Sigma_1 = \{\arg z = 0\}$, $\Sigma_2 = \{\arg z = 2\pi/3\}$, $\Sigma_3 = \{\arg z = \pi\}$, and $\Sigma_4 = \{\arg z = -2\pi/3\}$ with orientation as shown in Figure 1. As usual in RH problems, the orientation defines a $+$ and a $-$ side on each part of the contour, where the $+$-side is on the left when traversing the contour according to its orientation. For a function f on $\mathbb{C} \setminus \Sigma \equiv \Omega$, we use $f_\pm$ to denote its limiting values on Σ taken from the $\pm$-side, provided such limiting values exist. The contour Σ divides the complex plane into four sectors Ω_j also shown in the figure.

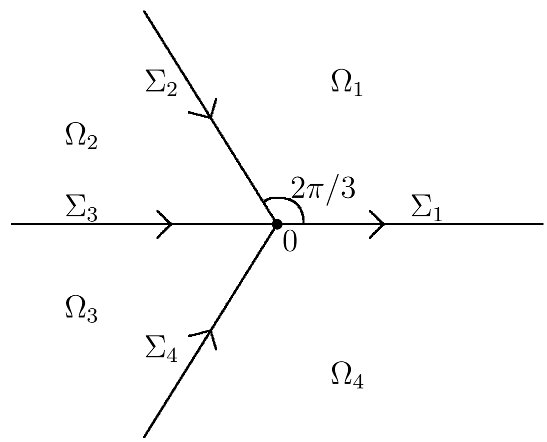

Figure 1: Contours for the model Riemann-Hilbert problem.

The model RH problem reads as follows.

Riemann-Hilbert problem for Ψ_α

(a) $\Psi_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_1$,

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_2,$$

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ for } z \in \Sigma_3,$$

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_4.$$

(c) $\Psi_\alpha(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(\frac{2}{3}z^{3/2} + sz^{1/2})\sigma_3}$ as $z \to \infty$.

Here $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the third Pauli matrix.

(d) $\Psi_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}$ as $z \to 0$, if $-1/2 < \alpha < 0$; and

$$\Psi_\alpha(z) = \begin{cases} O\begin{pmatrix} |z|^\alpha & |z|^{-\alpha} \\ |z|^\alpha & |z|^{-\alpha} \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in \Omega_1 \cup \Omega_4, \\ O\begin{pmatrix} |z|^{-\alpha} & |z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in \Omega_2 \cup \Omega_3, \end{cases} \quad \text{if } \alpha \geq 0.$$

Here, and in what follows, the O-terms are taken entrywise. Note that the RH problem depends on a parameter s through the asymptotic condition at infinity. If we want to emphasize the dependence on s we will write $\Psi_\alpha(z; s)$ instead of $\Psi_\alpha(z)$.

The model RH problem is uniquely solvable for every $\alpha > -1/2$ and $s \in \mathbb{R}$ (for details see [15, Proposition 2.1]).

All solutions of (1.3) are meromorphic in the complex plane. The special solution of relevance in [15] is characterized by the following result.

Theorem 1.1 *Assume $\alpha > -1/2$. Let $\Psi_\alpha(z; s)$ be the solution of the model RH problem and write*

$$\Psi_\alpha(z; s) = \left(I + \frac{m_\Psi(s)}{z} + O\left(\frac{1}{z^2}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(\frac{2}{3}z^{3/2} + sz^{1/2})\sigma_3} \tag{1.4}$$

as $z \to \infty$. Then

$$u_\alpha(s) = -\frac{s}{2} - i\frac{d}{ds}\left[m_\Psi(s)\right]_{12} \tag{1.5}$$

exists and satisfies (1.3)*. The function* (1.5) *is a global solution of* (1.3) *(i.e., it does not have poles on the real line).*

Moreover, u_α is also given by

$$u_\alpha(s) = i \lim_{z \to 0} \left[z \left(\frac{d}{dz}\Psi_\alpha(z)\right) \Psi_\alpha^{-1}(z)\right]_{12}. \tag{1.6}$$

Proof. The expression (1.6) follows from Lemma 3.2 in [15]. The remaining statements of Theorem 1.1 are contained in Theorem 1.4 of [15]. □

1.2 Main result

The aim of the present paper is an analysis of the asymptotic behavior of the special solution $u_\alpha(s)$, described in Theorem 1.1, as $s \to \pm\infty$. Our main result is the following.

Theorem 1.2 *Let $u_\alpha(s)$ be the solution of* (1.3) *given in Theorem 1.1. Then,*

$$u_\alpha(s) = \frac{\alpha}{\sqrt{s}} + O(s^{-2}), \qquad \text{as } s \to +\infty, \tag{1.7}$$

and

$$u_\alpha(s) = \frac{\alpha}{\sqrt{-s}} \cos\left(\frac{4}{3}(-s)^{3/2} - \alpha\pi\right) + O(s^{-2}), \qquad \text{as } s \to -\infty. \tag{1.8}$$

In the Sections 2 – 3, we provide a proof of our main result, Theorem 1.2. This is accomplished by using the Deift-Zhou steepest descent method for RH problems [7]. In the case at hand it consists of constructing a sequence of invertible transformations $\Psi_\alpha \mapsto A_\alpha \mapsto B_\alpha \mapsto C_\alpha \mapsto D_\alpha$, where the matrix-valued function D_α is close to the identity as $s \to \pm\infty$. By following the above transformations asymptotics for Ψ_α and thus, in view of (1.5) and (1.6), for $u_\alpha(s)$ may be derived.

1.3 RH problem for Painlevé XXXIV

In this paper we are mainly concerned with the special solution $u_\alpha(s)$. The analysis of the general solution of the Painlevé XXXIV equation (1.3) can be also performed via the nonlinear steepest descent method applied to the following generalization of the RH problem above.

Riemann-Hilbert problem for the general solution of PXXXIV

(a) $\Psi_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}$, for $z \in \Sigma_1$,

$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix}$, for $z \in \Sigma_2$,

$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \Sigma_3$,

$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ b_4 & 1 \end{pmatrix}$, for $z \in \Sigma_4$.

(c) $\Psi_\alpha(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-(\frac{2}{3}z^{3/2} + sz^{1/2})\sigma_3}$ as $z \to \infty$.

(d) If $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, then

$$\Psi_\alpha(z) = B(z) \begin{pmatrix} z^\alpha & 0 \\ 0 & z^{-\alpha} \end{pmatrix} E_j, \qquad \text{for } z \in \Omega_j, \tag{1.9}$$

where B is analytic. If $\alpha \in \frac{1}{2} + \mathbb{N}_0$, then there exists a constant κ such that

$$\Psi_\alpha(z) = B(z) \begin{pmatrix} z^\alpha & \kappa z^\alpha \log z \\ 0 & z^{-\alpha} \end{pmatrix} E_j, \qquad \text{for } z \in \Omega_j, \tag{1.10}$$

where B is analytic.

The complex numbers b_1, b_2, b_4 (the Stokes multipliers) and the constant invertible matrices E_j, $j = 1, 2, 3, 4$ (the connection matrices) form the RH data. They satisfy certain general constraints which in particular yield the following *cyclic* relation for the parameters b_j,

$$b_1 + b_2 + b_4 - b_1 b_2 b_4 = 2\cos(2\alpha\pi). \tag{1.11}$$

Except for the special case,

$$2\alpha = n, \quad b_1 = b_2 = b_4 = (-1)^n, \quad n \in \mathbb{N}, \tag{1.12}$$

when the solution of the RH problem is given in fact in terms of the Airy functions, the connection matrices E_j are determined (up to inessential left diagonal or upper triangular factors) by α and the Stokes multipliers b_j [1].

In the formulation of the general RH problem we keep the previous notation $\Psi_\alpha(z)$ for its solution. Formulas (1.5) and (1.6) for the solution of equation (1.3) are still valid, although, of course, the function $u_\alpha(s)$ for an arbitrary choice of the monodromy parameters b_j might have poles on the real line.

The case of our special interest in this paper corresponds to the choice,

$$b_1 = 1, \quad b_2 = e^{2\alpha\pi i}, \quad b_4 = e^{-2\alpha\pi i}, \tag{1.13}$$

of the Stokes parameters b_j. In Section 2 we treat the case $s \to +\infty$, which turns out to be the easier case. In Section 3 we deal with the more involved case $s \to -\infty$. The main technical issue is the necessity to construct an extra, in comparison with the $+\infty$ case, parametrix with Bessel functions.

It is well known (see e.g. [11]) that the Painlevé XXXIV equation (1.3) can be in fact transformed to the Painlevé II equation. We discuss in detail some aspects of this transformation, relevant to our analysis, in the Appendix. In particular, we notice that the second Painlevé function which is associated with the special Painlevé XXXIV solution $u_\alpha(s)$ we are studying here is *not* the familiar in random matrix [22], [23], [3] and string [21] theories Hastings-McLeod function. In addition, we show that, although the asymptotic behavior of $u_\alpha(s)$ as $s \to +\infty$ can be extracted from the already known asymptotics of the second Painlevé transcendents, the asymptotic behavior of $u_\alpha(s)$ as $s \to -\infty$ needs indeed a separate analysis.

In the Appendix we also discuss the question of the uniqueness of the solution $u_\alpha(s)$. In fact, we show that there is a one-parameter family of solutions of equation (1.3) with the asymptotics (1.7). This family is characterizes by the choice,

$$b_2 = e^{2\alpha\pi i}, \quad b_4 = e^{-2\alpha\pi i}, \quad b_1 = \text{arbitrary complex number}, \tag{1.14}$$

of the monodromy data b_j. At the same time, we conjecture that the asymptotic condition (1.8) fixes the solution uniquely. We present the arguments in support of this conjecture

[1] We refer to [10, Chapter 5] for more details on the setting of the general RH problems for Painlevé equations.

which are based on certain observations related to the asymptotic investigation of the general Painlevé XXXIV RH problem.

We conclude the introduction by mentioning that the Painlevé XXXIV equation has also appeared in several other physical applications. In fact, in the already mentioned paper [21] it was the Painlevé XXXIV image of the Hastings-McLeod Painlevé II solution which showed up and not the solution itself.

2 Proof of Theorem 1.2: asymptotics as $s \to +\infty$

2.1 First transformation $\Psi_\alpha \mapsto A_\alpha$

Introduce

$$A_\alpha(z;s) = s^{\sigma_3/4}\Psi_\alpha(sz;s), \qquad z \in \mathbb{C} \setminus \Sigma. \tag{2.1}$$

It is then easy to see that A_α satisfies a RH problem similar to that for Ψ_α. In the following we often suppress the s-dependence of functions whenever they are understood.

Riemann-Hilbert problem for A_α

(a) $A_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) A_α has the same jumps on Σ as that of Ψ_α.

(c) $A_\alpha(z) = (I + O(1/z))z^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-s^{3/2}(\frac{2}{3}z^{3/2}+z^{1/2})\sigma_3}$ as $z \to \infty$.

(d) A_α has the same behavior near 0 as that of Ψ_α.

From (1.6) and (2.1) it follows that

$$u_\alpha(s) = \frac{i}{\sqrt{s}} \lim_{z\to 0} \left[z \left(\frac{d}{dz} A_\alpha(z) \right) A_\alpha^{-1}(z) \right]_{12}. \tag{2.2}$$

2.2 Second transformation $A_\alpha \mapsto B_\alpha$

Let us put

$$t = s^{3/2}, \tag{2.3}$$

which is the large parameter in the RH problem. Consider Figure 2.

Define

$$B_\alpha(z) = \begin{cases} A_\alpha(z), & \text{for } z \in I \cup III \cup IV \cup VI, \\ A_\alpha(z) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}, & \text{for } z \in II, \\ A_\alpha(z) \begin{pmatrix} 1 & 0 \\ -e^{-2\alpha\pi i} & 1 \end{pmatrix}, & \text{for } z \in V. \end{cases} \tag{2.4}$$

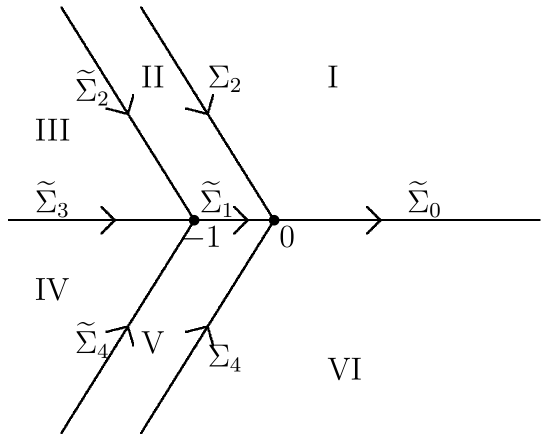

Figure 2: Contours and domains for the definition of B_α.

Let $\widetilde{\Sigma} = \Sigma - 1$, i.e., $\widetilde{\Sigma}$ is Σ translated to the left by one. Then

$$\widetilde{\Sigma} = \bigcup_{j=0}^{4} \widetilde{\Sigma}_j, \tag{2.5}$$

where the disjoint contours $\widetilde{\Sigma}_j$ are oriented as in Figure 2.

Riemann-Hilbert problem for B_α

(a) $B_\alpha : \mathbb{C} \setminus \widetilde{\Sigma} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $B_{\alpha,+}(z) = B_{\alpha,-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_0 = (0, \infty)$,

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \begin{pmatrix} e^{2\alpha\pi i} & 1 \\ 0 & e^{-2\alpha\pi i} \end{pmatrix}, \text{ for } z \in \widetilde{\Sigma}_1 = (-1, 0),$$

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}, \text{ for } z \in \widetilde{\Sigma}_2,$$

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ for } z \in \widetilde{\Sigma}_3 = (-\infty, -1),$$

$$B_{\alpha,+}(z) = B_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} & 1 \end{pmatrix}, \text{ for } z \in \widetilde{\Sigma}_4.$$

(c) $B_\alpha(z) = (I + O(1/z)) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-t(\frac{2}{3}z^{3/2} + z^{1/2})\sigma_3}$ as $z \to \infty$.

(d) If $\alpha < 0$, then $B_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}$ as $z \to 0$.

If $\alpha \geq 0$, then $B_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^{-\alpha} \\ |z|^\alpha & |z|^{-\alpha} \end{pmatrix}$ as $z \to 0$.

From (2.2) and (2.4) it follows that

$$u_\alpha(s) = \frac{i}{\sqrt{s}} \lim_{z\to 0} \left[z \left(\frac{d}{dz} B_\alpha(z) \right) B_\alpha^{-1}(z) \right]_{12}. \tag{2.6}$$

2.3 Third transformation $B_\alpha \mapsto C_\alpha$

We next introduce the g-function

$$g(z) = \frac{2}{3}(z+1)^{3/2}, \qquad -\pi < \arg(z+1) < \pi. \tag{2.7}$$

By a straightforward computation

$$g(z) - \left(\frac{2}{3} z^{3/2} + z^{1/2} \right) = \frac{1}{4} z^{-1/2} + O(z^{-3/2}), \qquad \text{as } z \to \infty. \tag{2.8}$$

Define

$$C_\alpha(z) = \begin{pmatrix} 1 & 0 \\ -it/4 & 1 \end{pmatrix} B_\alpha(z) e^{tg(z)\sigma_3}. \tag{2.9}$$

Then C_α satisfies the following RH problem.

Riemann-Hilbert problem for C_α

(a) $C_\alpha : \mathbb{C} \setminus \widetilde{\Sigma} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & e^{-2tg(z)} \\ 0 & 1 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_0$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} e^{2\alpha\pi i} & e^{-2tg(z)} \\ 0 & e^{-2\alpha\pi i} \end{pmatrix}$, for $z \in \widetilde{\Sigma}_1$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} e^{2tg(z)} & 1 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_2$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_3$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} e^{2tg(z)} & 1 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_4$.

(c) $C_\alpha(z) = (I + O(1/z)) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

(d) If $\alpha < 0$, then $C_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}$ as $z \to 0$.

If $\alpha \geq 0$, then $C_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^{-\alpha} \\ |z|^\alpha & |z|^{-\alpha} \end{pmatrix}$ as $z \to 0$.

Note that the prefactor $\left(\begin{smallmatrix} 1 & 0 \\ -it/4 & 1 \end{smallmatrix}\right)$ in the definition (2.9) of C_α is needed for the asymptotic condition (c) in the RH problem. The prefactor does not affect the 12 entry and so does not influence the computation of u_α via the formula (2.6).

Thus by (2.9)

$$\left[z\left(\frac{d}{dz}B_\alpha(z)\right)B_\alpha^{-1}(z)\right]_{12} = \left[z\left(\frac{d}{dz}C_\alpha(z)\right)C_\alpha^{-1}(z)\right]_{12} - tg'(z)\left[zC_\alpha(z)\sigma_3 C_\alpha^{-1}(z)\right]_{12}. \tag{2.10}$$

In view of item (d) in the RH problem for C_α (and the fact that $\det C_\alpha \equiv 1$) we have that $zC_\alpha(z)\sigma_3 C_\alpha^{-1}(z) \to 0$ as $z \to 0$. Therefore the second term in the right-hand side of (2.10) vanishes as $z \to 0$ and it follows by (2.6) that

$$u_\alpha(s) = \frac{i}{\sqrt{s}} \lim_{z\to 0}\left[z\left(\frac{d}{dz}C_\alpha(z)\right)C_\alpha^{-1}(z)\right]_{12}. \tag{2.11}$$

2.4 Construction of parametrices

2.4.1 Global parametrix $P_\alpha^{(\infty)}$

Away from the point -1 we expect that C_α should be well approximated by the solution $P_\alpha^{(\infty)}$ of the following RH problem.

Riemann-Hilbert problem for $P_\alpha^{(\infty)}$

(a) $P_\alpha^{(\infty)} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_{\alpha,+}^{(\infty)}(z) = P_{\alpha,-}^{(\infty)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in (-\infty, -1)$,

$P_{\alpha,+}^{(\infty)}(z) = P_{\alpha,-}^{(\infty)}(z)\begin{pmatrix} e^{2\alpha\pi i} & 0 \\ 0 & e^{-2\alpha\pi i} \end{pmatrix}$, for $z \in (-1, 0)$.

(c) $P_\alpha^{(\infty)}(z) = (I + O(1/z))z^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

It should be noted that $P_\alpha^{(\infty)}$ does not depend on s.

We seek $P_\alpha^{(\infty)}$ in the form

$$P_\alpha^{(\infty)}(z) = F_\alpha(z) z^{\alpha\sigma_3}, \tag{2.12}$$

where F_α is analytic in $\mathbb{C} \setminus (-\infty, -1]$. Clearly then the jump is correct on $(-1, 0)$. A straightforward computation shows that in order to have the correct jump also on $(-\infty, -1)$ we may take

$$F_\alpha(z) = E(z+1)^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}(\delta_\alpha(z))^{\sigma_3}, \tag{2.13}$$

where E is a constant prefactor and

$$\delta_\alpha(z) = \exp\left(-\frac{\alpha}{\pi}(z+1)^{1/2}\int_1^\infty \frac{\log t}{\sqrt{t-1}(t+z)}\,dt\right). \tag{2.14}$$

Using the residue theorem and a contour deformation argument, it is also straightforward to see that

$$\begin{aligned}\int_1^\infty \frac{\log t}{\sqrt{t-1}(t+z)}\,dt &= \frac{\pi\log z}{(z+1)^{1/2}} + \pi\int_0^1 \frac{dt}{\sqrt{1-t}(t+z)} \\ &= \frac{\pi\log z}{(z+1)^{1/2}} + \frac{\pi\ \log\left(\frac{(z+1)^{1/2}+1}{(z+1)^{1/2}-1}\right)}{(z+1)^{1/2}}.\end{aligned} \tag{2.15}$$

Hence,

$$\begin{aligned}P_\alpha^{(\infty)}(z) &= E(z+1)^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}(\delta_\alpha(z))^{\sigma_3}z^{\alpha\sigma_3} \\ &= E(z+1)^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\left(\frac{(z+1)^{1/2}+1}{(z+1)^{1/2}-1}\right)^{-\alpha\sigma_3}.\end{aligned} \tag{2.16}$$

In order to satisfy the asymptotic condition (c) of the RH problem we should take

$$E = \begin{pmatrix}1 & 0\\ 2\alpha i & 1\end{pmatrix}. \tag{2.17}$$

Note that

$$\frac{d}{dz}\log\left(\frac{(z+1)^{1/2}+1}{(z+1)^{1/2}-1}\right) = -\frac{1}{z(z+1)^{1/2}}$$

from which it follows after straightforward calculations from (2.16) and (2.17) that

$$\begin{aligned}\lim_{z\to 0} z\left(\frac{d}{dz}P_\alpha^{(\infty)}(z)\right)\left(P_\alpha^{(\infty)}(z)\right)^{-1} &= E\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}(\alpha\sigma_3)\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}E^{-1} \\ &= -i\alpha\begin{pmatrix}-2\alpha i & 1\\ -1+4\alpha^2 & 2\alpha i\end{pmatrix}.\end{aligned} \tag{2.18}$$

2.4.2 Local parametrix $P_\alpha^{(-1)}$

The global parametrix $P_\alpha^{(\infty)}$ will not be a good approximation to C_α near the point -1. Let $U^{(-1)}$ be a small open disc around -1 of radius < 1. We seek a local parametrix $P_\alpha^{(-1)}$ defined in $U^{(-1)}$ which satisfies the following.

Riemann-Hilbert problem for $P_\alpha^{(-1)}$

(a) $P_\alpha^{(-1)} : \overline{U^{(-1)}} \setminus \widetilde{\Sigma} \to \mathbb{C}^{2\times 2}$ is continuous and analytic on $U^{(-1)} \setminus \widetilde{\Sigma}$.

(b) $P_{\alpha,+}^{(-1)}(z) = P_{\alpha,-}^{(-1)}(z)\, v_{C_\alpha}(z)$ for $z \in \widetilde{\Sigma} \cap U^{(-1)}$, where v_{C_α} denotes the jump matrix for C_α (the contour having the same orientation as $\widetilde{\Sigma}$).

(c) $P_\alpha^{(-1)}(z) \left(P_\alpha^{(\infty)}(z)\right)^{-1} = I + O\left(\frac{1}{t}\right)$, as $t \to \infty$, uniformly for $z \in \partial U^{(-1)} \setminus \widetilde{\Sigma}$.

We seek $P_\alpha^{(-1)}$ in the form

$$P_\alpha^{(-1)}(z) = \widehat{P}_\alpha^{(-1)}(z) e^{tg(z)\sigma_3}, \tag{2.19}$$

where $\widehat{P}_\alpha^{(-1)}$ satisfies the following RH problem with constant jumps.

Riemann-Hilbert problem for $\widehat{P}_\alpha^{(-1)}$

(a) $\widehat{P}_\alpha^{(-1)} : \overline{U^{(-1)}} \setminus \widetilde{\Sigma} \to \mathbb{C}^{2\times 2}$ is continuous and analytic on $U^{(-1)} \setminus \Sigma_S$.

(b) $\widehat{P}_{\alpha,+}^{(-1)}(z) = \widehat{P}_{\alpha,-}^{(-1)}(z) \begin{pmatrix} e^{2\alpha\pi i} & 1 \\ 0 & e^{-2\alpha\pi i} \end{pmatrix}$, for $z \in \widetilde{\Sigma}_1 \cap U^{(-1)}$,

$\widehat{P}_{\alpha,+}^{(-1)}(z) = \widehat{P}_{\alpha,-}^{(-1)}(z) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_2 \cap U^{(-1)}$,

$\widehat{P}_{\alpha,+}^{(-1)}(z) = \widehat{P}_{\alpha,-}^{(-1)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_3 \cap U^{(-1)}$,

$\widehat{P}_{\alpha,+}^{(-1)}(z) = \widehat{P}_{\alpha,-}^{(-1)}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} & 1 \end{pmatrix}$, for $z \in \widetilde{\Sigma}_4 \cap U^{(-1)}$.

(c) $\widehat{P}_\alpha^{(-1)}(z) = P_\alpha^{(\infty)}(z)\Big(I + O\left(\frac{1}{t}\right)\Big) e^{-tg(z)\sigma_3}$, as $t \to \infty$, uniformly for $z \in \partial U^{(-1)} \setminus \widetilde{\Sigma}$.

A solution to this RH problem can be constructed in terms of Airy functions. The standard Airy parametrix is posed in an auxiliary ζ-plane and satisfies the following RH problem for a contour Σ as in Figure 1.

Riemann-Hilbert problem for $\Phi^{(Ai)}$

(a) $\Phi^{(Ai)} : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\Phi_+^{(Ai)} = \Phi_-^{(Ai)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, on Σ_1,

$\Phi_+^{(Ai)} = \Phi_-^{(Ai)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, on $\Sigma_2 \cup \Sigma_4$,

$\Phi_+^{(Ai)} = \Phi_-^{(Ai)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, on Σ_3.

(c) $\Phi^{(Ai)}(\zeta) = \zeta^{-\sigma_3/4}(I + O(\zeta^{-3/2}))\frac{1}{2\sqrt{\pi}}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}$ as $\zeta \to \infty$.

The solution is built out of the functions

$$y_0(\zeta) = \mathrm{Ai}(\zeta), \quad y_1(\zeta) = \omega \mathrm{Ai}(\omega\zeta), \quad y_2(\zeta) = \omega^2 \mathrm{Ai}(\omega^2\zeta), \qquad \omega = e^{2\pi i/3},$$

and takes the following form

$$\begin{cases} \Phi^{(Ai)} = \begin{pmatrix} -y_1 & -y_2 \\ -y_1' & -y_2' \end{pmatrix} \text{ in } \Omega_2, & \Phi^{(Ai)} = \begin{pmatrix} y_0 & -y_2 \\ y_0' & -y_2' \end{pmatrix} \text{ in } \Omega_1, \\ \Phi^{(Ai)} = \begin{pmatrix} -y_2 & y_1 \\ -y_2' & y_1' \end{pmatrix} \text{ in } \Omega_3, & \Phi^{(Ai)} = \begin{pmatrix} y_0 & y_1 \\ y_0' & y_1' \end{pmatrix} \text{ in } \Omega_4. \end{cases} \tag{2.20}$$

Then we put

$$\Phi_\alpha^{(Ai)}(\zeta) = \sqrt{2\pi}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\Phi^{(Ai)}(\zeta)e^{\pm\alpha\pi i\sigma_3}, \qquad \text{for } \pm \operatorname{Im}\zeta > 0, \tag{2.21}$$

and $\Phi_\alpha^{(Ai)}$ satisfies the following RH problem.

Riemann-Hilbert problem for $\Phi_\alpha^{(Ai)}$

(a) $\Phi_\alpha^{(Ai)} : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\Phi_{\alpha,+}^{(Ai)} = \Phi_{\alpha,-}^{(Ai)}\begin{pmatrix} e^{2\alpha\pi i} & 1 \\ 0 & e^{-2\alpha\pi i} \end{pmatrix}$, on Σ_1,

$\Phi_{\alpha,+}^{(Ai)} = \Phi_{\alpha,-}^{(Ai)}\begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}$, on Σ_2,

$\Phi_{\alpha,+}^{(Ai)} = \Phi_{\alpha,-}^{(Ai)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, on Σ_3,

$\Phi_{\alpha,+}^{(Ai)} = \Phi_{\alpha,-}^{(Ai)}\begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} & 1 \end{pmatrix}$, on Σ_4.

(c) $\Phi_\alpha^{(Ai)}(\zeta) = \zeta^{-\sigma_3/4}\Big(I + O(\zeta^{-3/2})\Big)\dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{\pm\alpha\pi i\sigma_3}e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}$

as $\zeta \to \infty$ with $\pm\operatorname{Im}\zeta > 0$.

Define

$$\widehat{P}_\alpha^{(-1)}(z) = E_\alpha(z)\Phi_\alpha^{(Ai)}(s(z+1)), \quad \text{for } z \in U^{(-1)} \setminus \widetilde{\Sigma}, \tag{2.22}$$

where E_α is analytic in $U^{(-1)}$. Then $\widehat{P}_\alpha^{(-1)}$ has the correct jumps. In order to satisfy the matching condition in the RH problem we take E_α in the following way:

$$E_\alpha(z) = P_\alpha^{(\infty)}(z)e^{\mp\alpha\pi i\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(s(z+1))^{\sigma_3/4}. \tag{2.23}$$

It is a straightforward computation to verify that E_α extends as an analytic function in $U^{(-1)}$. Combining (2.19), (2.22), and (2.23), we see that

$$P_\alpha^{(-1)}(z) = P_\alpha^{(\infty)}(z) e^{\mp\alpha\pi i\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (s(z+1))^{\sigma_3/4} \Phi_\alpha^{(Ai)}(s(z+1)) e^{tg(z)\sigma_3},$$

which completes the construction of the local parametrix $P_\alpha^{(-1)}$.

2.5 Fourth transformation $C_\alpha \mapsto D_\alpha$

Define now the final transformation

$$D_\alpha(z) = \begin{cases} C_\alpha(z)\big(P_\alpha^{(-1)}(z)\big)^{-1}, & \text{for } z \in U^{(-1)} \setminus \widetilde{\Sigma}, \\ C_\alpha(z)\big(P_\alpha^{(\infty)}(z)\big)^{-1}, & \text{for } z \in \mathbb{C} \setminus (\overline{U^{(-1)} \cup \widetilde{\Sigma}}). \end{cases} \tag{2.24}$$

Since C_α and $P_\alpha^{(\infty)}$ have the same jumps on $(-\infty,-1) \setminus U^{(-1)}$, and C_α and $P_\alpha^{(-1)}$ have the same jumps on $U^{(-1)} \cap \widetilde{\Sigma}$, we have that D_α is analytic across these contours. What remains are jumps for D_α on the contour Σ_D shown in Figure 3. Indeed, D_α satisfies the following RH problem.

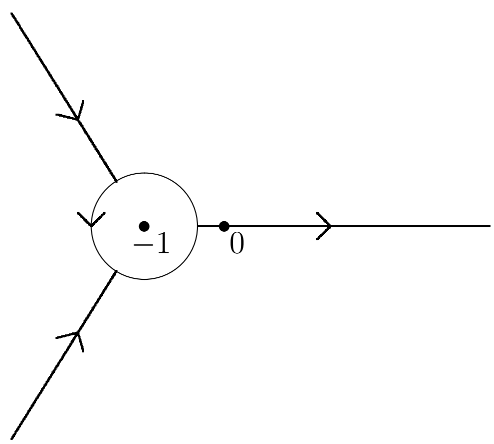

Figure 3: Contour Σ_D in the RH problem for D_α.

Riemann-Hilbert problem for D_α

(a) $D_\alpha : \mathbb{C} \setminus \Sigma_D \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $D_{\alpha,+}(z) = D_{\alpha,-}(z)\, v_{D_\alpha}(z)$ for $z \in \Sigma_D$, where

$$v_{D_\alpha} = \begin{cases} P_\alpha^{(\infty)}\,(P_\alpha^{(-1)})^{-1}, & \text{on } \partial U^{(-1)}, \\ P_{\alpha,-}^{(\infty)}\, v_{C_\alpha}\,(P_{\alpha,+}^{(\infty)})^{-1}, & \text{on } \Sigma_D \setminus \partial U^{(-1)}. \end{cases} \tag{2.25}$$

(c) $D_\alpha(z) = I + O(1/z)$ as $z \to \infty$.

The subscripts $\pm$ in $P^{(\infty)}$ are only relevant for the segment of the horizontal part of the contour to the left of 0.

2.6 Conclusion of the proof of (1.7)

The jump matrix satisfies

$$v_{D_\alpha}(z) = I + O(1/t), \quad \text{as } t \to \infty, \tag{2.26}$$

uniformly on the circle $\partial U^{(-1)}$. In addition

$$v_{D_\alpha}(z) = I + O(e^{-ct(|z|+1)}), \quad c > 0, \tag{2.27}$$

uniformly on $\Sigma_D \setminus \partial U^{(-1)}$. In a standard way (see e.g. [15]) one shows that

$$D_\alpha(z) = I + O\left(\frac{1}{t(1+|z|)}\right), \quad \text{as } t \to \infty, \tag{2.28}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_D$.

Finally, we have by (2.24) and the fact that $D_\alpha(z)$ and $\frac{d}{dz}D_\alpha(z)$ remain bounded as $z \to 0$,

$$\lim_{z\to\infty} z \left(\frac{d}{dz}C_\alpha(z)\right) C_\alpha^{-1}(z) = D_\alpha(0) \lim_{z\to\infty} z \left(\frac{d}{dz}P_\alpha^{(\infty)}(z)\right) \left(P_\alpha^{(\infty)}(z)\right)^{-1} D_\alpha^{-1}(0)$$

so that in view of (2.11) and (2.18)

$$u_\alpha(s) = \frac{\alpha}{\sqrt{s}} \left[D_\alpha(0) \begin{pmatrix} -2\alpha i & 1 \\ -1+4\alpha^2 & 2\alpha i \end{pmatrix} D_\alpha^{-1}(0)\right]_{12}. \tag{2.29}$$

Inserting (2.28) with $z = 0$ into (2.29) and recalling that $t = s^{3/2}$, we obtain (1.7).

3 Proof of Theorem 1.2: asymptotics as $s \to -\infty$

For the asymptotics as $s \to -\infty$ we also perform a sequence of transformations of the model RH problem $\Psi_\alpha \mapsto A_\alpha \mapsto B_\alpha \mapsto C_\alpha \mapsto D_\alpha$, but the transformations are different from the ones we performed for $s \to +\infty$. Thus A_α, B_α, C_α and D_α will now have a different meaning which hopefully does not lead to any confusion. We assume throughout this section that $s < 0$.

3.1 First transformation $\Psi_\alpha \mapsto A_\alpha$

Similar to (2.1), we introduce

$$A_\alpha(z; s) = (-s)^{\sigma_3/4}\Psi_\alpha(-sz; s), \qquad z \in \mathbb{C} \setminus \Sigma. \tag{3.1}$$

The A_α - RH problem reads as follows (the s-dependence is, as usual, suppressed).

Riemann-Hilbert problem for A_α

(a) $A_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps of A_α on Σ are the same as those of Ψ_α.

(c) $A_\alpha(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-t(\frac{2}{3}z^{3/2} - z^{1/2})\sigma_3}$ as $z \to \infty$.

(d) A_α has the same behavior near 0 as Ψ_α has.

Here, the large positive parameter t is defined by the equation (cf. (2.3))

$$t = (-s)^{3/2}. \tag{3.2}$$

Using (1.6) together with (3.1) we can express u_α in terms of A_α as follows

$$u_\alpha(s) = \frac{i}{\sqrt{-s}} \lim_{z\to 0} \left[z \left(\frac{d}{dz} A_\alpha(z) \right) A_\alpha^{-1}(z) \right]_{12}. \tag{3.3}$$

3.2 Second transformation $A_\alpha \mapsto B_\alpha$

An important difference comparing with the previous case is that a step analogous to the B_α - step is skipped. That is, our next step will be the g-function “dressing”.

Put (cf.(2.7))

$$g(z) = \frac{2}{3}(z-1)^{3/2}, \quad -\pi < \arg(z-1) < \pi. \tag{3.4}$$

Note that, as before,

$$g(z) - \left(\frac{2}{3} z^{3/2} - z^{1/2} \right) = \frac{1}{4} z^{-1/2} + O(z^{-3/2}), \qquad \text{as } z \to \infty. \tag{3.5}$$

Define

$$B_\alpha(z) = \begin{pmatrix} 1 & 0 \\ -\frac{it}{4} & 1 \end{pmatrix} A_\alpha(z) e^{tg(z)\sigma_3}. \tag{3.6}$$

Then, B_α satisfies the following RH problem.

Riemann-Hilbert problem for B_α

(a) $B_\alpha : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $B_{\alpha,+}(z) = B_{\alpha,-}(z)\begin{pmatrix} 1 & e^{-2tg(z)} \\ 0 & 1 \end{pmatrix}$, for $z \in (1,\infty)$,

$B_{\alpha,+}(z) = B_{\alpha,-}(z)\begin{pmatrix} e^{-t(g_-(z)-g_+(z))} & 1 \\ 0 & e^{t(g_-(z)-g_+(z))} \end{pmatrix}$, for $z \in (0,1)$,

$B_{\alpha,+}(z) = B_{\alpha,-}(z)\begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i+2tg(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_2$,

$B_{\alpha,+}(z) = B_{\alpha,-}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \Sigma_3$,

$B_{\alpha,+}(z) = B_{\alpha,-}(z)\begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i+2tg(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_4$.

(c) $B_\alpha(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

(d) The behavior of $B_\alpha(z)$ as $z \to 0$ is the same as that of $\Psi_\alpha(z)$.

We emphasize, that the contour Σ is now *the same* as in the original Ψ_α - problem.

From the transformation (3.6) it follows that

$$\left[\frac{d}{dz}A_\alpha(z)A_\alpha^{-1}(z)\right]_{12} = \left[\frac{d}{dz}B_\alpha(z)B_\alpha^{-1}(z)\right]_{12} - tg'(z)\left[B_\alpha(z)\sigma_3 B_\alpha^{-1}(z)\right]_{12}. \tag{3.7}$$

From part (d) in the RH problem satisfied by B_α we can deduce that

$$B_\alpha(z)\sigma_3 B_\alpha^{-1}(z) = \begin{cases} O(|z|^{2\alpha}) & \text{if } -1/2 < \alpha < 0, \\ O(1) & \text{if } \alpha \geq 0, \end{cases}$$

as $z \to 0$. It follows that we can forget about the second term in the right-hand side of (3.7) and we obtain from (3.3) and (3.7) that

$$u_\alpha(s) = \frac{i}{\sqrt{-s}}\lim_{z\to 0}\left[z\left(\frac{d}{dz}B_\alpha(z)\right)B_\alpha^{-1}(z)\right]_{12}. \tag{3.8}$$

3.3 Third transformation $B_\alpha \mapsto C_\alpha$

In order to proceed further, we need to analyze the structure of the sign of $\operatorname{Re} g(z)$. The first observation is trivial:

$$g(z) = \frac{2}{3}|z-1|^{3/2} \geq 0, \quad z \in (1,\infty). \tag{3.9}$$

Next, we notice that the function $w = g(z)$ performs a conformal mapping of the upper half plane to the sector $0 < \arg w < 3\pi/2$. Under this mapping, the domain $\frac{\pi}{3} < \arg(z-1) < \pi$ becomes the left half plane $\operatorname{Re} w < 0$, and the ray Σ_2 transforms to a simple smooth curve

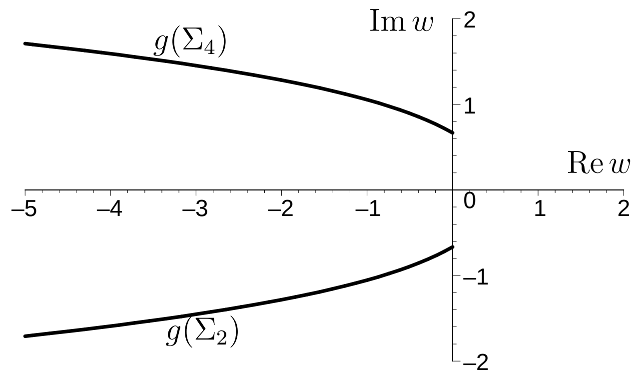

Figure 4: Images of the rays Σ_2 and Σ_4 under the mapping $z \mapsto w = g(z) = \frac{2}{3}(z-1)^{3/2}$.

$g(\Sigma_2)$ which ends at $-2i/3$, lies entirely in the left half plane, and behaves for large $z \in \Sigma_2$ as

$$\operatorname{Re} w \sim -\frac{2}{3}|z|^{3/2}, \qquad \operatorname{Im} w \sim -\frac{1}{2}\sqrt{3}|z|^{1/2},$$

see Figure 4. Similarly, the function $w = g(z)$ performs a conformal mapping of the lower half plane to the sector $-3\pi/2 < \arg w < 0$. Under this mapping, the domain $-\pi < \arg(z-1) < -\frac{\pi}{3}$ becomes the left half plane $\operatorname{Re} w < 0$, and the ray Σ_4 is mapped to the mirror image of $g(\Sigma_2)$ with respect to the real axis, see again Figure 4. Therefore, there exists a constant $c > 0$ such that

$$\operatorname{Re} g(z) \le -c|z-1| \le 0, \quad z \in \Sigma_2 \cup \Sigma_4. \tag{3.10}$$

We also notice that the function,

$$h(z) := g_-(z) - g_+(z), \quad z \in (0,1), \tag{3.11}$$

admits analytic continuation into the domains Ω_u and Ω_d indicated in Figure 5. Indeed we have,

$$h(z) = -2g(z), \quad z \in \Omega_u, \tag{3.12}$$

and

$$h(z) = 2g(z), \quad z \in \Omega_d. \tag{3.13}$$

The indicated above characterization of the conformal mapping generated by the function $g(z)$ yields the inequalities

$$\operatorname{Re} h(z) > 0, \quad z \in \Omega_u, \tag{3.14}$$

and

$$\operatorname{Re} h(z) < 0, \quad z \in \Omega_d. \tag{3.15}$$

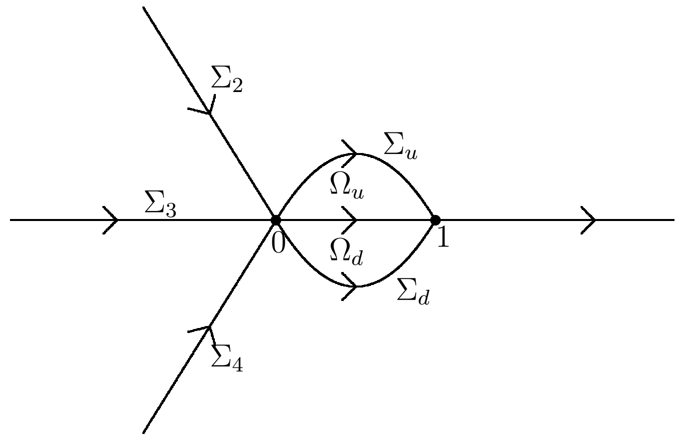

Figure 5: Contour $\Sigma_C = \Sigma \cup \Sigma_u \cup \Sigma_d$ and domains for the definition of C_α.

The estimates (3.9) and (3.10) imply that the jump matrices on the rays $(1,\infty)$, Σ_2 and Σ_4 and away of the end points are close to the identity matrix, while the estimates (3.14) and (3.15) suggest to "open the lenses" around the interval $(0,1)$.

Noticing that

$$\begin{aligned} \begin{pmatrix} e^{-t(g_-(z)-g_+(z))} & 1 \\ 0 & e^{t(g_-(z)-g_+(z))} \end{pmatrix} &= \begin{pmatrix} e^{-th(z)} & 1 \\ 0 & e^{th(z)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e^{th(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-th(z)} & 1 \end{pmatrix}, \end{aligned} \tag{3.16}$$

we define the new function C_α with the help of the following equations.

$$C_\alpha(z) = \begin{cases} B_\alpha(z), & \text{for } z \notin \Omega_u \cup \Omega_d, \\ B_\alpha(z) \begin{pmatrix} 1 & 0 \\ -e^{-th(z)} & 1 \end{pmatrix}, & \text{for } z \in \Omega_u, \\ B_\alpha(z) \begin{pmatrix} 1 & 0 \\ e^{th(z)} & 1 \end{pmatrix}, & \text{for } z \in \Omega_d. \end{cases} \tag{3.17}$$

We use Σ_u and Σ_d to denote the curves which, in conjunction with the interval $[0,1]$, make the boundary of the lenses Ω_u and Ω_d, respectively. The curves Σ_u and Σ_d together with their orientation are indicated in Figure 5. Let Σ_C denote the contour Σ augmented by the arcs Σ_u and Σ_d. Then, C_α satisfies the following RH problem.

Riemann-Hilbert problem for C_α

(a) $C_\alpha : \mathbb{C} \setminus \Sigma_C \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & e^{-2tg(z)} \\ 0 & 1 \end{pmatrix}$, for $z \in (1,\infty)$,

$$C_{\alpha,+}(z) = C_{\alpha,-}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ for } z \in (0,1)\cup\Sigma_3,$$

$$C_{\alpha,+}(z) = C_{\alpha,-}(z)\begin{pmatrix} 1 & 0 \\ e^{2tg(z)} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_u\cup\Sigma_d,$$

$$C_{\alpha,+}(z) = C_{\alpha,-}(z)\begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i+2tg(z)} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_2,$$

$$C_{\alpha,+}(z) = C_{\alpha,-}(z)\begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i+2tg(z)} & 1 \end{pmatrix}, \text{ for } z \in \Sigma_4.$$

(c) $C_\alpha(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

(d) $C_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}$ as $z \to 0$, if $-1/2 < \alpha < 0$; and

$$C_\alpha(z) = \begin{cases} O\begin{pmatrix} |z|^\alpha & |z|^{-\alpha} \\ |z|^\alpha & |z|^{-\alpha} \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in (\Omega_1\cup\Omega_4)\setminus(\Omega_u\cup\Omega_d), \\ O\begin{pmatrix} |z|^{-\alpha} & |z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in \Omega_2\cup\Omega_3\cup\Omega_u\cup\Omega_d, \end{cases} \quad \text{if } \alpha \geq 0.$$

When formulating the jump conditions across the lenses boundaries, i.e., on the curves Σ_u and Σ_d, we have replaced the function $h(z)$ by the function $g(z)$ according to the relations (3.12) and (3.13). The contours for the C_α - RH problem are depicted in Figure 5.

To express u_α in terms of C_α we can use the same formula (3.8) but with the understanding that $z \to 0$ from outside the lens. Thus

$$u_\alpha(s) = \frac{i}{\sqrt{-s}} \lim_{\substack{z\to 0 \\ z\notin\Omega_u\cup\Omega_d}} \left[z\left(\frac{d}{dz}C_\alpha(z)\right)C_\alpha^{-1}(z)\right]_{12}. \tag{3.18}$$

3.4 Construction of parametrices

3.4.1 Global parametrix $P^{(\infty)}$

Away from the points 0 and 1 the jump matrices on $(1,\infty)$, Σ_2, Σ_4, Σ_u, and Σ_d all tend to the identity matrix as $t \to +\infty$ at an exponential rate. Therefore, away from the points 0 and 1 we expect that C_α should be well approximated by the solution $P^{(\infty)}$ of the following RH problem with the only nontrivial jump across the ray $(-\infty, 1)$.

Riemann-Hilbert problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C}\setminus(-\infty,1] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P_+^{(\infty)}(z) = P_-^{(\infty)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in (-\infty, 1)$.

(c) $P^{(\infty)}(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

It should be noted that now $P^{(\infty)}$ is clearly independent of both s and α.

This Riemann-Hilbert problem is even simpler than the corresponding problem for the case of positive s, and its solution is obviously given by the formula (cf. (2.16))

$$P^{(\infty)}(z) = (z-1)^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad -\pi < \arg(z-1) < \pi. \tag{3.19}$$

Near the points 0 and 1 the parametrix $P^{(\infty)}$ cannot be expected to represent the asymptotics of C_α. Our next task is to construct the parametrix solutions near the points mentioned.

3.4.2 Local parametrix $P^{(1)}$

We begin with the point $z = 1$. We will see that the form of parametrix at this point is very similar to the parametrix near the point $z = -1$ in the previous case of positive s.

Let $U^{(1)}$ be a small open disc around 1. We seek a parametrix $P^{(1)}$ defined in $U^{(1)}$ which satisfies the following RH problem.

Riemann-Hilbert problem for $P^{(1)}$

(a) $P^{(1)} : \overline{U^{(1)}} \setminus \Sigma_C \to \mathbb{C}^{2\times 2}$ is continuous and analytic on $U^{(1)} \setminus \Sigma_C$.

(b) $P^{(1)}_+(z) = P^{(1)}_-(z)\, v_{C_\alpha}(z)$ for $z \in \Sigma_C \cap U^{(1)}$.

(c) $P^{(1)}(z) \left(P^{(\infty)}(z)\right)^{-1} = I + O\left(\frac{1}{t}\right)$ as $t \to \infty$, uniformly for $z \in \partial U^{(1)} \setminus \Sigma_C$.

Similar to the case of positive s, we use v_{C_α} to denote the jump matrix in the RH problem for C_α. Comparing this Riemann-Hilbert problem with the one for the function $P^{(-1)}_\alpha$ from the previous section, we see that the solution $P^{(1)}$ can be given in terms of the matrix function $\Phi^{(Ai)}_\alpha$ defined by equation (2.21) and evaluated at $\alpha = 0$. Indeed, we propose the following form for the function $P^{(1)}$ (cf. (2.19), (2.22)) :

$$P^{(1)}(z) = (-s)^{\sigma_3/4} \Phi^{(Ai)}_0(-s(z-1)) e^{tg(z)\sigma_3}, \quad \text{for } z \in U^{(1)} \setminus \Sigma_C, \tag{3.20}$$

where $\Phi^{(Ai)}_0 := \Phi^{(Ai)}_{\alpha=0}$. We note that the relevant left multiplier $E(z)$ is chosen to be a scaling factor $(-s)^{\sigma_3/4}$, and that this is exactly what the matrix function $E_\alpha(z)$ from (2.23) reduces to if $\alpha = 0$ and if we replace $s(z+1)$ by $-s(z-1)$ as is appropriate in the present situation.

Defined by (3.20), the function $P^{(1)}(z)$ has obviously the correct jumps. Moreover, when z belongs to the boundary of $U^{(1)}$ and $-s$ is large, the function $\Phi_0(-s(z-1))$ in the r.h.s. of (3.20) can be replaced by its asymptotics. Therefore, we have:

$$\begin{aligned} P^{(1)}(z) &= (z-1)^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + O\left(\frac{1}{(-s)^{3/2}}\right)\right) \\ &= P^{(\infty)}(z)\Big(I + O\left(\frac{1}{t}\right)\Big), \end{aligned} \tag{3.21}$$

as $t \to \infty$, uniformly for $z \in \partial U^{(1)} \setminus \Sigma_C$, which yields the matching condition needed. The parametrix in the neighborhood of the point $z = 1$ is then constructed. Note that it does not depend on α.

3.5 Local parametrix $P_\alpha^{(0)}$

We are passing now to the analysis of the neighborhood of the point $z = 0$. Let $U^{(0)}$ be a small open disc around 0. We seek a parametrix $P_\alpha^{(0)}$ defined in $U^{(0)}$ which satisfies the following RH problem.

Riemann-Hilbert problem for $P_\alpha^{(0)}$

(a) $P_\alpha^{(0)} : \overline{U^{(0)}} \setminus \Sigma_C \to \mathbb{C}^{2\times 2}$ is continuous and analytic on $U^{(0)} \setminus \Sigma_C$.

(b) $P_{\alpha,+}^{(0)}(z) = P_{\alpha,-}^{(0)}(z)\, v_{C_\alpha}(z)$ for $z \in \Sigma_C \cap U^{(0)}$.

(c) $P_\alpha^{(0)}(z) \left(P^{(\infty)}(z)\right)^{-1} = I + O\left(\frac{1}{t}\right)$ as $t \to \infty$, uniformly for $z \in \partial U^{(0)} \setminus \Sigma_C$.

(d) $P_\alpha^{(0)}$ has the same behavior near 0 as C_α.

The contours and the jump matrices in the RH problem for $P_\alpha^{(0)}$ are depicted in Figure 6.

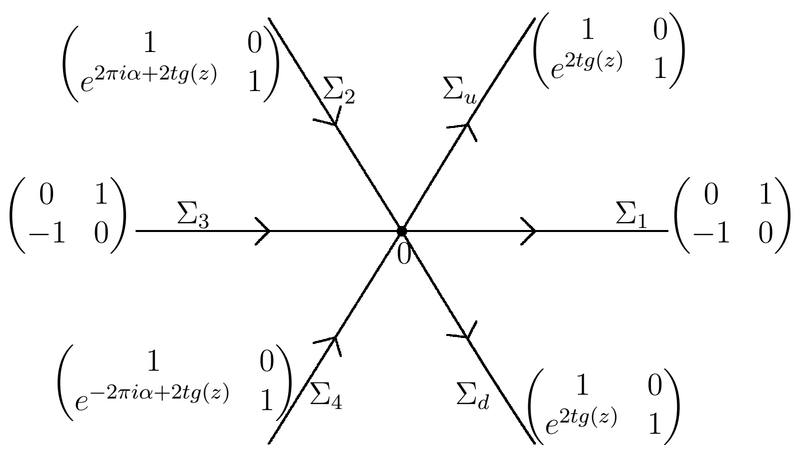

Figure 6: Contours and jump matrices for the RH problem for $P_\alpha^{(0)}$ (magnified picture).

We take $P_\alpha^{(0)}$ in the form

$$P_\alpha^{(0)}(z) = \widehat{P}_\alpha^{(0)}(z) e^{tg(z)\sigma_3} \tag{3.22}$$

and we see that $\widehat{P}_\alpha^{(0)}$ should satisfy a RH problem with jumps that are indicated in Figure 7. The jump matrices for $\widehat{P}_\alpha^{(0)}$ are constant along the six different pieces.

We now first construct the solution $\Phi_\alpha^{(Bes)}$ of a model RH problem with the same constant jumps on six infinite rays in an auxiliary ζ-plane, and then we put

$$\widehat{P}_\alpha^{(0)}(z) = E_\alpha(z)\Phi_\alpha^{(Bes)}(tf(z)), \tag{3.23}$$

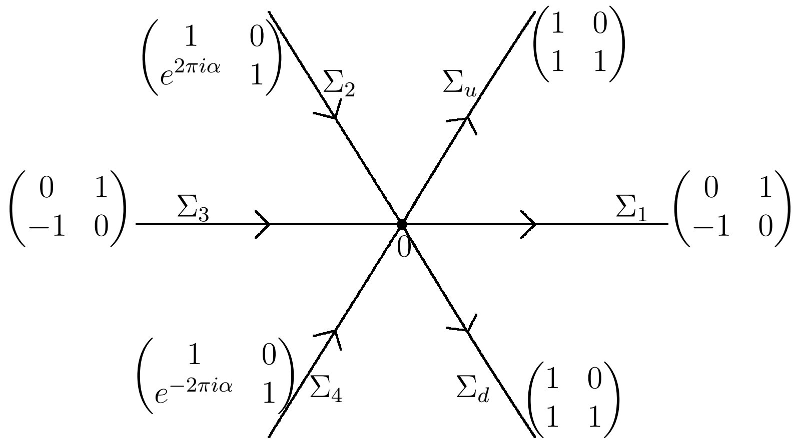

Figure 7: Contours and jump matrices for the RH problem for $\widehat{P}_\alpha^{(0)}$ (magnified picture).

where f is given by

$$f(z) = \frac{2}{3} - \frac{2i}{3}(z-1)^{3/2}, \quad 0 < \arg(z-1) < 2\pi, \tag{3.24}$$

and $E_\alpha(z)$ is an analytic prefactor that will be chosen later. The function $f(z)$ is analytic in $U^{(0)}$. Moreover, $f(z) = z + \cdots$ and therefore it defines a conformal map in the neighborhood $U^{(0)}$. After performing a slight contour deformation, we may and do assume that the six contours Σ_1, Σ_2, Σ_3, Σ_4, Σ_u and Σ_d are mapped by f into six rays. The exact relation between the functions $f(z)$ and $g(z)$ is given by the formula

$$f(z) = \frac{2}{3} - i \begin{cases} g(z), & \operatorname{Im} z > 0, \\ -g(z), & \operatorname{Im} z < 0. \end{cases} \tag{3.25}$$

We construct $\Phi_\alpha^{(Bes)}$ by relating it to a model RH problem constructed by Vanlessen [24] and also used in [20], whose solution we denote here by $\widetilde{\Phi}_\alpha^{(Bes)}$.

Riemann-Hilbert problem for $\widetilde{\Phi}_\alpha^{(Bes)}$

(a) $\widetilde{\Phi}_\alpha^{(Bes)} : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is analytic, where Γ is the union of the eight half rays shown in Figure 8, namely

$$\Gamma = \{\zeta \in \mathbb{C} \mid \arg \zeta \in \{0, \pm\pi/3, \pm\pi/2, \pm 2\pi/3, \pi\}\}.$$

(b) $\widetilde{\Phi}_{\alpha,+}^{(Bes)} = \widetilde{\Phi}_{\alpha,-}^{(Bes)} v_{\widetilde{\Phi}_\alpha^{(Bes)}}$ on Γ, where the constant jump matrices $v_{\widetilde{\Phi}_\alpha^{(Bes)}}$ are indicated in Figure 8.

(d) $\widetilde{\Phi}_\alpha^{(Bes)}(\zeta) = O\begin{pmatrix} |\zeta|^\alpha & |\zeta|^\alpha \\ |\zeta|^\alpha & |\zeta|^\alpha \end{pmatrix}$ as $\zeta \to 0$, if $-1/2 < \alpha < 0$; and

$$\widetilde{\Phi}_\alpha^{(Bes)}(\zeta) = \begin{cases} O\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{-\alpha} \end{pmatrix} & \text{as } \zeta \to 0,\ \zeta \in \mathbb{C}\setminus\Gamma \text{ with } \pi/3 < |\arg\zeta| < 2\pi/3, \\ O\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \end{pmatrix} & \text{as } \zeta \to 0,\ \zeta \in \mathbb{C}\setminus\Gamma \text{ with } \zeta \text{ elsewhere,} \end{cases}$$

if $\alpha \geq 0$.

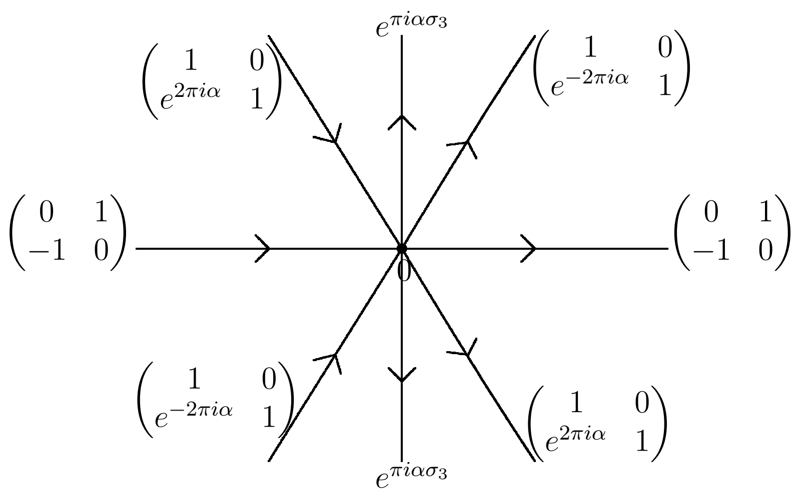

Figure 8: Contours and jump matrices for the RH problem for $\widetilde{\Phi}_\alpha^{(Bes)}$.

We did not specify the asymptotic condition. A solution of the RH problem for $\widetilde{\Phi}_\alpha^{(Bes)}$ was given in terms of Bessel functions of orders $\alpha \pm \frac{1}{2}$. There is a different expression in each sector. Let it suffice here to mention the solution in the sector $\pi/2 < \arg\zeta < 2\pi/3$. We give it in a form that is different from the one in [24], where the modified Bessel functions $I_{\alpha\pm\frac{1}{2}}$ and $K_{\alpha\pm\frac{1}{2}}$ are used. We state it here in terms of the usual Bessel functions $J_{\alpha\pm\frac{1}{2}}$ and the Hankel function of first kind $H^{(1)}_{\alpha\pm\frac{1}{2}}$ as follows

$$\widetilde{\Phi}_\alpha^{(Bes)}(\zeta) = \sqrt{\pi} e^{-\frac{1}{4}\pi i} \zeta^{1/2} \begin{pmatrix} J_{\alpha+\frac{1}{2}}(\zeta) & \frac{1}{2} H^{(1)}_{\alpha+\frac{1}{2}}(\zeta) \\ J_{\alpha-\frac{1}{2}}(\zeta) & \frac{1}{2} H^{(1)}_{\alpha-\frac{1}{2}}(\zeta) \end{pmatrix}, \qquad \pi/2 < \arg\zeta < 2\pi/3. \quad (3.26)$$

The asymptotics as $\zeta \to \infty$ in this sector is

$$\widetilde{\Phi}_\alpha^{(Bes)}(\zeta) = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} + \frac{\alpha}{2\zeta} \begin{pmatrix} -i(\alpha+1) & \alpha+1 \\ -(\alpha-1) & i(\alpha-1) \end{pmatrix} + O\left(\frac{1}{\zeta^2}\right) \right] \times e^{\frac{1}{4}\pi i\sigma_3} e^{\frac{1}{2}\pi i\alpha\sigma_3} e^{-i\zeta\sigma_3}, \qquad \pi/2 < \arg\zeta < 2\pi/3. \quad (3.27)$$

Then we define

$$\Phi_\alpha^{(Bes)}(\zeta) = e^{-(\frac{1}{2}\pi i\alpha + \frac{1}{4}\pi i)\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \widetilde{\Phi}_\alpha^{(Bes)}(\zeta) \times \begin{cases} e^{\pi i\alpha\sigma_3}, & \text{if } \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta > 0, \\ I & \text{if } \operatorname{Re}\zeta < 0, \\ e^{-\pi i\alpha\sigma_3}, & \text{if } \operatorname{Re}\zeta > 0,\ \operatorname{Im}\zeta < 0. \end{cases} \quad (3.28)$$

It is then easy to check that $\Phi_\alpha^{(Bes)}$ is analytic across $i\mathbb{R}$ and has the jump matrices indicated in Figure 7, but of course on the contour $\Gamma \setminus i\mathbb{R}$. The behavior at 0 remains unaffected by the above transformation, while the constant prefactor $e^{-(\frac{1}{2}\pi i\alpha+\frac{1}{4}\pi i)\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$ is chosen so that we obtain the precise asymptotics as $\zeta \to \infty$ given in item (c) below.

Riemann-Hilbert problem for $\Phi_\alpha^{(Bes)}$

(a) $\Phi_\alpha^{(Bes)} : \mathbb{C} \setminus (\Gamma \setminus i\mathbb{R}) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\Phi_{\alpha,+}^{(Bes)} = \Phi_{\alpha,-}^{(Bes)} v_{\Phi_\alpha^{(Bes)}}$ on $\Gamma \setminus i\mathbb{R}$, where the constant jump matrices $v_{\Phi_\alpha^{(Bes)}}$ are indicated in Figure 7.

(c) $\Phi_\alpha^{(Bes)}(\zeta) = \left(I + O\left(\frac{1}{\zeta}\right)\right) e^{-i\zeta\sigma_3}$ as $\zeta \to \infty$ with $\operatorname{Im}\zeta > 0$, and

$$\Phi_\alpha^{(Bes)}(\zeta)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \left(I + O\left(\frac{1}{\zeta}\right)\right) e^{-i\zeta\sigma_3} \text{ as } \zeta \to \infty \text{ with } \operatorname{Im}\zeta < 0.$$

(d) $\Phi_\alpha^{(Bes)}(\zeta) = O\begin{pmatrix}|\zeta|^\alpha & |\zeta|^\alpha\\ |\zeta|^\alpha & |\zeta|^\alpha\end{pmatrix}$ as $\zeta \to 0$, if $-1/2 < \alpha < 0$; and

$$\Phi_\alpha^{(Bes)}(\zeta) = \begin{cases} O\begin{pmatrix}|\zeta|^\alpha & |\zeta|^{-\alpha}\\ |\zeta|^\alpha & |\zeta|^{-\alpha}\end{pmatrix} & \text{as } \zeta \to 0,\ \zeta \in \mathbb{C}\setminus\Gamma \text{ with } \pi/3 < |\arg\zeta| < 2\pi/3,\\ O\begin{pmatrix}|\zeta|^{-\alpha} & |\zeta|^{-\alpha}\\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha}\end{pmatrix} & \text{as } \zeta \to 0,\ \zeta \in \mathbb{C}\setminus\Gamma \text{ with } \zeta \text{ elsewhere,}\end{cases}$$

if $\alpha \geq 0$.

As noted before, we now put

$$P_\alpha^{(0)}(z) = E_\alpha(z)\Phi_\alpha^{(Bes)}(tf(z))e^{tg(z)\sigma_3},$$

where E_α is still to be determined. Then, for fixed $z \in \partial U^{(0)}$ we have as $t \to \infty$,

$$\begin{aligned} P_\alpha^{(0)}(z) &= E_\alpha(z)\Phi_\alpha^{(Bes)}(tf(z))e^{tg(z)\sigma_3}\\ &= \begin{cases} E_\alpha(z)\left(I + O(1/t)\right)e^{-2it\sigma_3/3}, & \operatorname{Im} z > 0,\\ E_\alpha(z)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\left(I + O(1/t)\right)e^{2it\sigma_3/3}, & \operatorname{Im} z < 0,\end{cases}\\ &= \begin{cases} E_\alpha(z)e^{-2it\sigma_3/3}\left(I + O(1/t)\right), & \operatorname{Im} z > 0,\\ E_\alpha(z)e^{-2it\sigma_3/3}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\left(I + O(1/t)\right), & \operatorname{Im} z < 0.\end{cases} \end{aligned} \tag{3.29}$$

To match this with $P^{(\infty)}(z)$ for $z \in \partial U^{(0)}$ we choose

$$E_\alpha(z) = \begin{cases} P^{(\infty)}(z)e^{2it\sigma_3/3}, & \operatorname{Im} z > 0,\\ P^{(\infty)}(z)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}e^{2it\sigma_3/3}, & \operatorname{Im} z < 0,\end{cases} \tag{3.30}$$

which is indeed analytic in $U^{(0)}$. This completes the construction of the local parametrix $P_\alpha^{(0)}$.

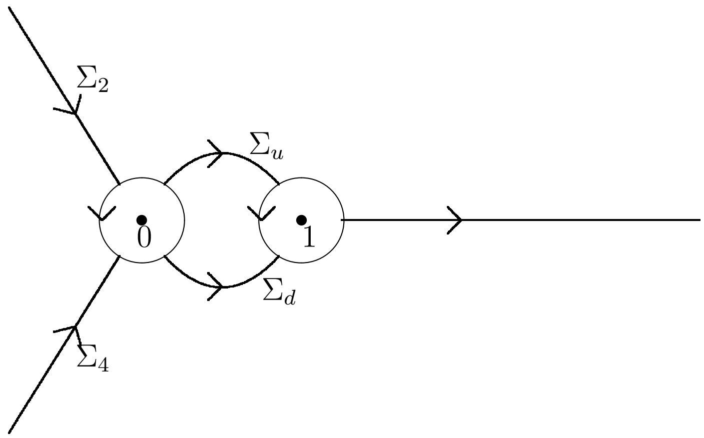

Figure 9: Contour Σ_D in the RH problem for D_α.

3.6 Fourth transformation $C_\alpha \mapsto D_\alpha$

In the final transformation we put

$$D_\alpha(z) = \begin{cases} C_\alpha(z)\left(P^{(1)}(z)\right)^{-1}, & \text{for } z \in U^{(1)} \setminus \Sigma_C, \\ C_\alpha(z)\left(P_\alpha^{(0)}(z)\right)^{-1}, & \text{for } z \in U^{(0)} \setminus \Sigma_C, \\ C_\alpha(z)\left(P^{(\infty)}(z)\right)^{-1}, & \text{for } z \in \mathbb{C} \setminus (\overline{U^{(1)} \cup U^{(0)} \cup (-\infty, 1)}). \end{cases} \tag{3.31}$$

By construction, the only jumps that remain for D_α are across the circles $\partial U^{(1)}$ and $\partial U^{(0)}$ and the parts of the arcs Σ_u and Σ_d and the rays $[1,\infty)$, Σ_2, and Σ_4 which lie outside of the neighborhoods $U^{(1)}$ and $U^{(0)}$. We shall denote this remaining contour as Σ_D; it is depicted in Figure 9. The Riemann-Hilbert problem for D_α is set on this contour.

Riemann-Hilbert problem for D_α

(a) $D_\alpha : \mathbb{C} \setminus \Sigma_D \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $D_{\alpha,+}(z) = D_{\alpha,-}(z)\, v_{D_\alpha}(z)$ for $z \in \Sigma_D$, where

$$v_{D_\alpha} = \begin{cases} P^{(\infty)}\,(P^{(1)})^{-1}, & \text{on } \partial U^{(1)}, \\ P^{(\infty)}\,(P^{(0)})^{-1}, & \text{on } \partial U^{(0)}, \\ P^{(\infty)}\, v_{C_\alpha}\,(P^{(\infty)})^{-1}, & \text{on } \Sigma_D \setminus (\partial U^{(1)} \cup \partial U^{(0)}). \end{cases} \tag{3.32}$$

(c) $D_\alpha(z) = I + O(1/z)$ as $z \to \infty$.

Due to the matching conditions of the Riemann-Hilbert problems for $P^{(1)}$ and $P^{(0)}$, we have that

$$v_{D_\alpha}(z) = I + O\left(\frac{1}{t}\right), \tag{3.33}$$

uniformly on the circles $\partial U^{(1)}$ and $\partial U^{(0)}$. Simultaneously,

$$v_{D_\alpha}(z) = I + O\left(e^{-ct|z|}\right), \quad c > 0, \tag{3.34}$$

uniformly on $\Sigma_D \setminus (\partial U^{(1)} \cup \partial U^{(0)})$. Hence, as before,

$$D_\alpha(z) = I + O\Big(\frac{1}{t(1+|z|)}\Big), \qquad \text{as } t \to +\infty, \tag{3.35}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_D$.

3.7 Conclusion of the proof of (1.8)

The main remaining step in the proof of (1.8) is to express u_α in terms of D_α. The result of the calculations is contained in the next lemma.

Lemma 3.1 *For every $s < 0$, we have*

$$u_\alpha(s) = \frac{\alpha}{\sqrt{-s}} \left[D_\alpha(0) \begin{pmatrix} i\sin\theta & \cos\theta \\ -\cos\theta & -i\sin\theta \end{pmatrix} D_\alpha^{-1}(0) \right]_{12}. \tag{3.36}$$

where

$$\theta = 4t/3 - \pi\alpha. \tag{3.37}$$

and $t = (-s)^{3/2}$ as before.

Theorem 1.2 follows immediately from the lemma and (3.35). Indeed from (3.35) we find that $D_\alpha(0) = I + O((-s)^{-3/2})$ as $s \to -\infty$. Hence by (3.36) and (3.37)

$$u_\alpha(s) = \frac{\alpha}{\sqrt{-s}} \left(\cos\theta + O((-s)^{-3/2})\right) = \frac{\alpha}{\sqrt{-s}} \cos\left(\frac{4}{3}(-s)^{3/2} - \pi\alpha\right) + O(s^{-2})$$

as $s \to -\infty$, which is (1.8).

So it remains to prove Lemma 3.1.

Proof. Take $z \in U^{(0)}$ with $\operatorname{Im} z > 0$ and outside of the lense. Then we have by (3.31), (3.22), and (3.23), that

$$\begin{aligned} C_\alpha(z) &= D_\alpha(z) P_\alpha^{(0)}(z) \\ &= D_\alpha(z) E_\alpha(z) \Phi_\alpha^{(Bes)}(tf(z)) e^{tg(z)\sigma_3}. \end{aligned} \tag{3.38}$$

In the computation of the limit of $z \left[\left(\frac{d}{dz} C_\alpha(z)\right) C_\alpha^{-1}(z)\right]_{12}$ as $z \to 0$, the only term that will contribute is the one we get by taking the derivative of $\Phi_\alpha^{(Bes)}(tf(z))$. This easily follows from the fact that D_α and E_α are analytic at 0.

Since, by (3.30) and (3.19), we have

$$E_\alpha(0) = e^{-i\pi\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}e^{2it\sigma_3/3} \tag{3.39}$$

it then follows from (3.18) and (3.38) that

$$\begin{aligned} u_\alpha(s) =& \frac{i}{\sqrt{-s}}\Bigg[D_\alpha(0)e^{-i\pi\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}e^{2it\sigma_3/3}\\ &\times\left(\lim_{z\to 0} z\left(\frac{d}{dz}\Phi_\alpha^{(Bes)}(tf(z))\right)\left(\Phi_\alpha^{(Bes)}(tf(z))\right)^{-1}\right)\\ &\times e^{-2it\sigma_3/3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}e^{i\pi\sigma_3/4}D_\alpha^{-1}(0)\Bigg]_{12}. \end{aligned} \tag{3.40}$$

Putting $\zeta = tf(z)$ we get

$$\frac{d}{dz}\Phi_\alpha^{(Bes)}(tf(z)) = tf'(z)\frac{d}{d\zeta}\Phi_\alpha^{(Bes)}(\zeta).$$

Noting that $f(0) = 0$ and $f'(0) \neq 0$, we find

$$\frac{tf'(z)z}{\zeta} = \frac{zf'(z)}{f(z)} \to 1$$

as $z \to 0$. Therefore

$$\lim_{z\to 0} z\left(\frac{d}{dz}\Phi_\alpha^{(Bes)}(tf(z))\right)\left(\Phi_\alpha^{(Bes)}(tf(z))\right)^{-1} = \lim_{\zeta\to 0}\zeta\left(\frac{d}{d\zeta}\Phi_\alpha^{(Bes)}(\zeta)\right)\left(\Phi_\alpha^{(Bes)}(\zeta)\right)^{-1}. \tag{3.41}$$

From the definition (3.28) of $\Phi_\alpha^{(Bes)}$ we find that for $\operatorname{Im}\zeta > 0$,

$$\begin{aligned} &\left(\frac{d}{d\zeta}\Phi_\alpha^{(Bes)}(\zeta)\right)\left(\Phi_\alpha^{(Bes)}(\zeta)\right)^{-1}\\ &= e^{-(\pi i\alpha/2+\pi i/4)\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\left(\left(\frac{d}{d\zeta}\widetilde{\Phi}_\alpha^{(Bes)}(\zeta)\right)\left(\widetilde{\Phi}_\alpha^{(Bes)}(\zeta)\right)^{-1}\right)\\ &\qquad\times\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}e^{(\pi i\alpha/2+\pi i/4)\sigma_3}. \end{aligned} \tag{3.42}$$

Recall that $\widetilde{\Phi}_\alpha^{(Bes)}$ is built out of Bessel functions. From the differential-difference relations satisfied by the Bessel functions (see formula 9.1.27 in [1])

$$\begin{aligned} J_\nu'(\zeta) &= J_{\nu-1}(\zeta) - \frac{\nu}{\zeta}J_\nu(\zeta),\\ J_{\nu-1}'(\zeta) &= -J_{\nu+1}(\zeta) + \frac{\nu}{\zeta}J_\nu(\zeta), \end{aligned}$$

and similar ones for the Hankel functions, it easily follows from (3.26) that

$$\frac{d}{d\zeta}\widetilde{\Phi}_{\alpha}^{(Bes)}(\zeta) = \begin{pmatrix} -\alpha/\zeta & 1 \\ -1 & \alpha/\zeta \end{pmatrix} \widetilde{\Phi}_{\alpha}^{(Bes)}(\zeta).$$

Thus

$$\lim_{\zeta\to 0} \zeta \left(\left(\frac{d}{d\zeta}\widetilde{\Phi}_{\alpha}^{(Bes)}(\zeta) \right) \left(\widetilde{\Phi}_{\alpha}^{(Bes)}(\zeta) \right)^{-1} \right) = -\alpha\sigma_3. \tag{3.43}$$

Combining (3.43) with (3.40), (3.41), (3.42) we get with $\theta = 4t/3 - \pi\alpha$,

$$\begin{aligned} u_\alpha(s) =& \frac{i}{\sqrt{-s}} \left[D_\alpha(0) e^{-i\pi\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{\theta i\sigma_3/2 - \pi i\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right. \\ & \left. \times \, (-\alpha\sigma_3) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{-\theta i\sigma_3/2 + \pi i\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} e^{i\pi\sigma_3/4} D_\alpha^{-1}(0) \right]_{12}, \end{aligned}$$

which after straightforward calculation reduces to (3.36). This completes the proof of the lemma. □

Acknowledgements

Alexander Its was supported in part by NSF grant #DMS-0401009. Arno Kuijlaars is supported by FWO-Flanders project G.0455.04, by K.U. Leuven research grants OT/04/21 and OT/08/33, by the Belgian Interuniversity Attraction Pole P06/02, by the European Science Foundation Program MISGAM, and by a grant from the Ministry of Science and Innovation of Spain, project code MTM2008-06689-C02-01. Jörgen Östensson is supported by K.U. Leuven research grant OT/04/21.

A Appendix. Relation to the Painlevé II equation and the uniqueness question

We start with reviewing the general facts concerning the relation between the thirty fourth and the second Painlevé equations.

Let $q(s)$ be a solution of the second Painlevé equation with parameter ν,

$$q'' = 2q^3 + sq - \nu. \tag{A.1}$$

Then, the function $u(s)$ defined by the formulae

$$u(s) = 2^{-1/3} U(-2^{1/3}s), \qquad U(s) = q^2(s) + q'(s) + \frac{s}{2}. \tag{A.2}$$

satisfies the thirty fourth equation (1.3) with the parameter

$$\alpha = \frac{\nu}{2} - \frac{1}{4}$$

(see [11]; see also [2] and [15]). The inverse transformation is given by the formulae

$$q(s) = -2^{-1/3}Q(-2^{-1/3}s), \qquad Q(s) = \frac{u' - 2\alpha}{2u}. \tag{A.3}$$

Moreover, the equations (see [2], [19]),

$$\Psi_\alpha(z;s) = \begin{pmatrix} 1 & 0 \\ \eta(s) & 1 \end{pmatrix} z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{\pi i \sigma_3/4} \Psi^{FN}_{2\alpha+1/2}(w; -2^{1/3}s) e^{-\pi i \sigma_3/4}, \tag{A.4}$$

$$\eta(s) = -2^{1/3}\left([m_{FN}(s)]_{11} + [m_{FN}(s)]_{12}\right),$$

where $w = e^{\pi i/2} 2^{-1/3} z^{1/2}$ with $\operatorname{Im} w > 0$, establish the relation between the solution $\Psi_\alpha(z;s)$ of the general Painlevé XXXIV RH problem formulated in Section 1.3 and the solution $\Psi^{FN}_\nu(w;s)$ of the RH problem associated with the Painlevé II equation [9] with the parameter $\nu = 2\alpha + 1/2$. In (A.4), $m_{FN}(s)$ denotes the first matrix coefficient of the expansion $\Psi^{FN}_\nu(w;s)$ at $w = \infty$,

$$\Psi^{FN}_\nu(w;s) = \left(I + \frac{m_{FN}(s)}{z} + O\left(\frac{1}{w^2}\right)\right) e^{-i(\frac{4}{3}w^3 + sw)\sigma_3} \tag{A.5}$$

as $w \to \infty$. We use here the Flaschka-Newell [9] form of the Painlevé II RH problem whose setting we will now remind (for details see [10, Chapter 5]).

The general Painlevé II RH problem involves three complex constants a_1, a_2, a_3 satisfying (cf. (1.11))

$$a_1 + a_2 + a_3 + a_1 a_2 a_3 = -2i \sin \nu\pi, \tag{A.6}$$

and certain connection matrices E^{FN}_j. Let $S_j = \{w \in \mathbb{C} \mid \frac{2j-3}{6}\pi < \arg w < \frac{2j-1}{6}\pi\}$ for $j = 1, \dots, 6$, and let $\Sigma^{FN} = \mathbb{C} \setminus \bigcup_j S_j$. Then Σ^{FN} consists of six rays $\Sigma^{FN}_j = \{w \in \mathbb{C} \mid \arg w = \frac{2j-1}{6}\pi\}$ for $j = 1, \dots, 6$, all chosen oriented towards infinity as in Figure 10.

The RH problem is the following.

(a) $\Psi^{FN}_\nu : \mathbb{C} \setminus \Sigma^{FN} \to \mathbb{C}^{2\times 2}$ is analytic,

(b) $\Psi^{FN}_{\nu,+} = \Psi^{FN}_{\nu,-} \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}$ on Σ^{FN}_1,

$\Psi^{FN}_{\nu,+} = \Psi^{FN}_{\nu,-} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$ on Σ^{FN}_2,

$\Psi^{FN}_{\nu,+} = \Psi^{FN}_{\nu,-} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix}$ on Σ^{FN}_3,

$\Psi^{FN}_{\nu,+} = \Psi^{FN}_{\nu,-} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}$ on Σ^{FN}_4,

$\Psi^{FN}_{\nu,+} = \Psi^{FN}_{\nu,-} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix}$ on Σ^{FN}_5,

$\Psi^{FN}_{\nu,+} = \Psi^{FN}_{\nu,-} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}$ on Σ^{FN}_6.

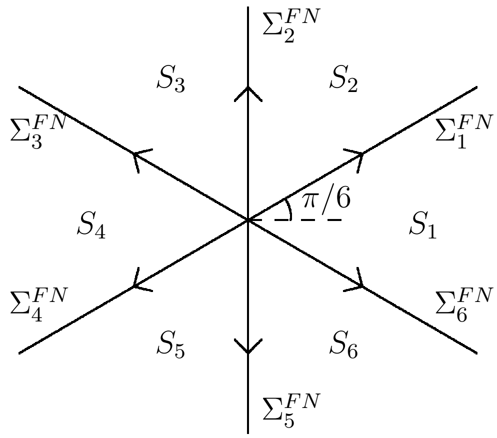

Figure 10: Contour for the RH problem for Ψ_ν^{FN}.

(c) $\Psi_\nu^{FN}(w) = (I + O(1/w))e^{-i(\frac{4}{3}w^3+sw)\sigma_3}$ as $w \to \infty$.

(d) If $\nu - \frac{1}{2} \notin \mathbb{N}_0$, then

$$\Psi_\nu^{FN}(w) = B(w) \begin{pmatrix} w^\nu & 0 \\ 0 & w^{-\nu} \end{pmatrix} E_j^{FN}, \quad \text{for } w \in S_j, \tag{A.7}$$

where B is analytic. If $\nu \in \frac{1}{2} + \mathbb{N}_0$, then there exists a constant κ such that

$$\Psi_\nu^{FN}(w) = B(w) \begin{pmatrix} w^\nu & \kappa w^\nu \log w \\ 0 & w^{-\nu} \end{pmatrix} E_j^{FN}, \quad \text{for } w \in S_j, \tag{A.8}$$

where B is analytic.

Except for the special case,

$$\nu = \frac{1}{2} + n, \quad a_1 = a_2 = a_3 = i(-1)^{n+1}, \quad n \in \mathbb{N}, \tag{A.9}$$

when the solution of the RH problem is given in fact in terms of the Airy functions, the connection matrices E_j^{FN} are determined (up to inessential left diagonal or upper triangular factors) by ν and the Stokes multipliers a_j. In the special case the solution is parametrized by the one non-trivial entry of the connection matrix E_1^{FN}. We refer to [10, Chapters 5 and 11] for more details on the setting and the analysis of the Painlevé II RH problem (see also our paper [15], where we review these results in the notations we use here).

Equation (A.4) implies the following relation between the Painlevé XXXIV and Painlevé II Stokes parameters ([2], [19]; see also [15]),

$$b_1 = ia_2, \quad b_2 = ia_3, \quad b_4 = ia_1, \tag{A.10}$$

where we use the b_j as in Section 1.3. Therefore, taking into account (1.13), we conclude that the second Painlevé function which is related to the special solution $u_\alpha(s)$ of the

thirty fourth Painlevé equation studied in this paper corresponds to the choice of the Stokes multipliers,

$$a_1 = e^{-\nu\pi i}, \quad a_2 = -i, \quad a_3 = -e^{\nu\pi i}. \tag{A.11}$$

This is different from the choice,

$$a_1 = e^{-\nu\pi i}, \quad a_2 = 0, \quad a_3 = -e^{\nu\pi i}, \tag{A.12}$$

corresponding, as it is shown in [10, Chapter 11] [2], to the generalized ($\nu \neq 0$) Hastings-McLeod solution of the second Painlevé equation, i.e., the solution which is characterized by the following asymptotic conditions,

$$q_{HM}(s) = \sqrt{-\frac{s}{2}} + O(s^{-1}), \qquad \text{as } s \to -\infty, \tag{A.13}$$

and

$$q_{HM}(s) = \frac{\nu}{s} + O(s^{-4}), \qquad \text{as } s \to +\infty, \tag{A.14}$$

see also [3].

It follows from (A.11), however, that both solutions - the Hastings-McLeod solution and the one corresponding to $u_\alpha(s)$, belong to the same one-parameter family of solutions of the Painlevé II equation which is characterizes by the following choice of the Stokes multipliers:

$$a_1 = e^{-\nu\pi i}, \quad a_3 = -e^{\nu\pi i}, \tag{A.15}$$

and the Stokes multiplier a_2 is a free parameter of the family. It is shown in [18] (see also [10, Chapter 11]) that this family is exactly the classical family of the so called *tronquée* solutions, i.e., the solutions all of which exhibit the same behavior (A.13) at $-\infty$. In fact, for every *tronquée* solution the behavior (A.13) can be extended to a full asymptotic series and it holds in the whole sector $2\pi/3 < \arg s < 4\pi/3$,

$$q_{tronq}(s) \sim \sqrt{-\frac{s}{2}} \sum_{n=0}^{\infty} c_n (-s)^{-3n/2}, \quad c_0 = 1,$$
$$\text{as } s \to -\infty, \quad \arg s \equiv \pi + \arg(-s) \in \left(\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right), \tag{A.16}$$

where all the coefficients c_n are uniquely determined by the substitution into the Painlevé II equation (i.e., the series is the *same* for every solution from the family [3]).

[2]The RH problem which is used in [10, Chapter 11] differs by a simple gauge transformation from the Flaschka-Newell RH problem. Indeed one has that $\Psi^{FN} = e^{-i\frac{\pi}{4}\sigma_3}\Psi e^{i\frac{\pi}{4}\sigma_3}$, where $\Psi(z)$ is the solution of the RH problem from [10, Chapter 11]. This in turn implies that the Flaschka-Newell monodromy parameters a_j are related to the monodromy parameters s_j from [10, Chapter 11], via the equations, $a_1 = is_1$, $a_2 = -is_2$, and $a_3 = is_3$.

[3]We refer to [10, Chapter 11] for more on the asymptotic analysis of the *tronquée* solutions of the second Painlevé equation. In particular, the reader can find there an alternative parametrization (and its explicit relation to a_2) of the solutions via the coefficients of the oscillatory terms of the asymptotics on the boundary rays.

The above made observation suggests that one can obtain the asymptotic statement (1.7) of Theorem 1.2 directly from (A.16) using relation (A.2). Indeed, the first two terms of (A.16) read

$$q_{tronq}(s) = \sqrt{-\frac{s}{2} - \frac{\nu}{2s} + O(s^{-5/2})},$$
$$\text{as } s \to -\infty, \quad \arg s \equiv \pi + \arg(-s) \in \left(\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right). \tag{A.17}$$

Substituting this (differentiable !) asymptotics into (A.2) we indeed arrive at (1.7).

The fact that the asymptotics (1.7) holds for a one-parameter family of the solutions of the Painlevé XXXIV equation (1.3) can also be deduced from the direct analysis of the Painlevé XXXIV RH problem. Denote the one-parameter family of solutions of Painlevé XXXIV corresponding to the RH data,

$$b_2 = e^{2\alpha\pi i}, \quad b_4 = e^{-2\alpha\pi i}, \quad b_1 = b \in \mathbb{C}, \tag{A.18}$$

as

$$u_\alpha^{(tronq)}(s) \equiv u_\alpha^{(tronq)}(s|b).$$

Note, that the cyclic relation (1.11) is valid identically for b_1 if b_2 and b_4 are as in (A.18). It is not difficult to see that *exactly* the same sequence of transformation as the one we used in Section 2 in the analysis of the RH problem in the case $s \to +\infty$ can be performed for any value of b. In the final D_α RH problem the only difference is in the jump matrix v_{D_α} on the segment of the horizontal part of the jump contour depicted in Figure 3 which is to the right of 0. That is, instead of

$$v_{D_\alpha} = P_\alpha^{(\infty)}(z) \begin{pmatrix} 1 & e^{-2tg(z)} \\ 0 & 1 \end{pmatrix} \left(P_\alpha^{(\infty)}(z)\right)^{-1} \equiv I + O\left(e^{-ct(|z|+1)}\right), \qquad \text{for} \quad z > 0,$$

we now have,

$$v_{D_\alpha} = P_\alpha^{(\infty)}(z) \begin{pmatrix} 1 & be^{-2tg(z)} \\ 0 & 1 \end{pmatrix} \left(P_\alpha^{(\infty)}(z)\right)^{-1} \equiv I + O\left(be^{-ct(|z|+1)}\right), \qquad \text{for} \quad z > 0.$$

In other words, the only difference in v_{D_α} is in the exponentially small error. Hence the estimates (2.26) and (2.27) are valid for all b and lead to the same asymptotic behavior (1.7) of the solution $u_\alpha^{(tronq)}(s)$ for all b. In fact, every solution from the family has the same asymptotic series representation,

$$u_\alpha^{(tronq)}(s) \sim \frac{\alpha}{\sqrt{s}} + \sum_{n=1}^{\infty} d_n s^{-\frac{3n+1}{2}}, \qquad \text{as } s \to +\infty, \tag{A.19}$$

with the coefficients d_n uniquely determined by the substitution of the series into equation (1.3) (see [15] for the explicit recurrence relation for d_n).

The asymptotic behavior (A.14) of the Hastings-McLeod solution at $+\infty$ is also shared by another one-parameter family of *tronquée* solutions. The corresponding RH parametrization is (see [13]; see also [10, Chapter 11]),

$$a_2 = 0, \quad a_1 + a_3 = -2i \sin \nu\pi. \tag{A.20}$$

The Painlevé II function which corresponds to the thirty fourth Painlevé function $u_\alpha(s)$ obviously does not belong to this family and hence does not behave as (A.14) when $s \to +\infty$. However, the leading term of its behavior as $s \to +\infty$ is known ([17]; see also [10, Chapter 10]). Unfortunately, the leading term is not enough to derive the corresponding asymptotics as $s \to -\infty$ of the Painlevé XXXIV function $u_\alpha(s)$. Indeed, the leading asymptotics of $q(s)$ as $s \to +\infty$ is of the form

$$q(s) \sim \sqrt{\frac{s}{2}} \cot\left(\frac{\sqrt{2}}{3} s^{3/2} + \chi\right), \tag{A.21}$$

(the phase χ is known) and it cancels out in the right-hand side of equation (A.2). Moreover, the solution $q(s)$, as it follows from (A.21), has poles on the positive real s-axis while the function $u_\alpha(s)$ is smooth for all real s. This means that the reduction to the Painlevé II equation is not the best way to study the asymptotics of the Painlevé XXXIV function $u_\alpha(s)$ as $s \to -\infty$. It is better to proceed via the direct analysis of the Painlevé XXXIV RH problem for Ψ_α, as we did in Section 3 of this paper [4].

Let us now consider the question of the uniqueness of the solution $u_\alpha(s)$. To this end, let us analyze what effect on the constructions of Section 3 would be produced by the passing to the general *tronquée* solution $u_\alpha^{(tronq)}(s)$, i.e., by lifting the restriction $b_1 = 1$. We already saw that the considerations and results of Section 2 are not affected. The situation with Section 3, i.e., with the analysis of the Painlevé XXXIV RH problem as $s \to -\infty$ is different. In what follows, we shall analyze the *tronquée*-RH problem assuming that

$$b > 0. \tag{A.22}$$

There are no changes in the basic three transformations of Section 3, except that the factorization (3.16) now is

$$\begin{pmatrix} e^{-th(z)} & b \\ 0 & e^{th(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b^{-1}e^{th(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b^{-1}e^{-th(z)} & 1 \end{pmatrix}, \tag{A.23}$$

and the transformation (3.17) is modified accordingly. It leads us to the following RH problem for the matrix function $C_\alpha(z)$.

(a) $C_\alpha : \mathbb{C} \setminus \Sigma_C \to \mathbb{C}^{2\times 2}$ is analytic.

[4]In principle, it is possible to use the inverse formula (A.3) and obtain the asymptotic of $u_\alpha(s)$ by integrating (A.21). However, with this approach we face the problem of evaluation of the constant of integration and, once again, one has to take special care of the poles of the Painlevé II function.

(b) $C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & be^{-2tg(z)} \\ 0 & 1 \end{pmatrix}$, for $z \in (1, \infty)$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$, for $z \in (0, 1)$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in \Sigma_3$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ b^{-1}e^{2tg(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_u \cup \Sigma_d$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i+2tg(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_2$,

$C_{\alpha,+}(z) = C_{\alpha,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i+2tg(z)} & 1 \end{pmatrix}$, for $z \in \Sigma_4$.

(c) $C_\alpha(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

(d) $C_\alpha(z) = O\begin{pmatrix} |z|^\alpha & |z|^\alpha \\ |z|^\alpha & |z|^\alpha \end{pmatrix}$ as $z \to 0$, if $-1/2 < \alpha < 0$; and

$$C_\alpha(z) = \begin{cases} O\begin{pmatrix} |z|^\alpha & |z|^{-\alpha} \\ |z|^\alpha & |z|^{-\alpha} \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in (\Omega_1 \cup \Omega_4) \setminus (\Omega_u \cup \Omega_d), \\ O\begin{pmatrix} |z|^{-\alpha} & |z|^{-\alpha} \\ |z|^{-\alpha} & |z|^{-\alpha} \end{pmatrix} & \text{as } z \to 0 \text{ with } z \in \Omega_2 \cup \Omega_3 \cup \Omega_u \cup \Omega_d, \end{cases} \quad \text{if } \alpha \geq 0.$$

The contour for this RH problem is the same as before, i.e., the one depicted in Figure 5.

We can at once make two important observations. Firstly, the neighborhood of the point $z = 0$ will contribute to the asymptotic analysis (as, in fact, it has in the case $b = 1$) and hence we should expect a change in the asymptotics (1.8) and appearance in it of an explicit dependence of the parameter b. Secondly, the inequality $b \neq 1$ yields serious alterations in the constructions of the parametrices $P^{(\infty)}$ and $P^{(0)}_\alpha$.

The RH problem for the global parametrix $P^{(\infty)}$ now reads.

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, 1] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}_+(z) = P^{(\infty)}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for $z \in (-\infty, 0)$,

$P^{(\infty)}_+(z) = P^{(\infty)}_-(z) \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$, for $z \in (0, 1)$,

(c) $P^{(\infty)}(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-\sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ as $z \to \infty$.

This RH problem still can be solved explicitly. In fact, it is now similar to the global parametrix from Section 2. The solution is given by the equation,

$$P^{(\infty)}(z) = E(z-1)^{-\sigma_3/4}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\left(\frac{(z-1)^{1/2}+i}{(z-1)^{1/2}-i}\right)^{\beta\sigma_3} \tag{A.24}$$

where

$$\beta = \frac{i}{2\pi}\log b, \qquad E = \begin{pmatrix}1 & 0\\ 2\beta & 1\end{pmatrix},$$

and the branches of the arguments are fixed by the inequalities,

$$-\pi < \arg(z-1) < \pi, \quad 0 < \arg\left(\frac{(z-1)^{1/2}+i}{(z-1)^{1/2}-i}\right) < \pi.$$

The construction of the local parametrix $P_\alpha^{(0)}$ now involves, instead of the Bessel model RH problem $\Phi_\alpha^{(Bes)}$, the function $\Phi_{\alpha,\beta}^{(CHF)}$ which satisfies a RH problem with the same contour Γ as $\Phi_\alpha^{(Bes)}$, and with the same jump matrices in the left half-plane, see Figure 7, while in the right half-plane one has the new jump matrices,

$$\begin{pmatrix}1 & 0\\ e^{2\pi i\beta} & 1\end{pmatrix}, \quad \begin{pmatrix}0 & e^{-2\pi i\beta}\\ -e^{2\pi i\beta} & 0\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}1 & 0\\ e^{2\pi i\beta} & 1\end{pmatrix},$$

on the rays $\arg\zeta = \pi/3$, $\arg\zeta = 0$, and $\arg\zeta = -\pi/3$, respectively. The jump contour is as shown in Figure 7, but on straight rays extending to infinity.

The function $\Phi_{\alpha,\beta}^{(CHF)}$, in turn, admits the following representation,

$$\Phi_{\alpha,\beta}^{(CHF)}(\zeta) = \widetilde{\Phi}_{\alpha,\beta}^{(CHF)}(\zeta)\begin{cases} e^{\left(\frac{\pi i\beta}{2}+\pi i\alpha\right)\sigma_3}, & \text{if } \operatorname{Re}\zeta > 0,\, \operatorname{Im}\zeta > 0,\\ e^{\frac{\pi i\beta}{2}\sigma_3} & \text{if } \operatorname{Re}\zeta < 0,\\ e^{\left(\frac{\pi i\beta}{2}-\pi i\alpha\right)\sigma_3}, & \text{if } \operatorname{Re}\zeta > 0,\, \operatorname{Im}\zeta < 0,\end{cases} \tag{A.25}$$

where the matrix function $\widetilde{\Phi}_{\alpha,\beta}^{(CHF)}(\zeta)$ is the solution of the RH problem depicted in Figure 11.

It is shown in [5] that the RH problem for $\widetilde{\Phi}_{\alpha,\beta}^{(CHF)}$ supplemented by the proper representation at $\zeta = 0$ (inherited from (1.9) and (1.10)) and the asymptotic condition,

$$\widetilde{\Phi}_{\alpha,\beta}^{(CHF)}(\zeta) = \left(I + O\left(\frac{1}{\zeta}\right)\right)\zeta^{-\beta\sigma_3}e^{-i\zeta\sigma_3} \qquad \text{as} \quad \zeta \to \infty, \quad 0 < \arg\zeta < \frac{\pi}{2}, \tag{A.26}$$

is uniquely solvable; moreover, it admits an explicit solution in terms of the confluent hypergeometric functions $\psi(a,c;\zeta)$ with the parameters,

$$a = \alpha + \beta, \qquad c = 1 + 2\alpha.$$

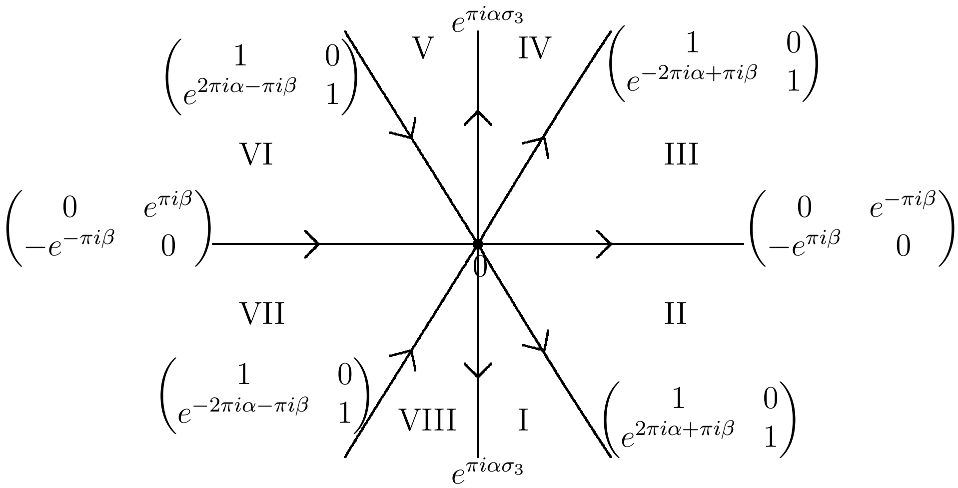

Figure 11: Contours and jump matrices for the RH problem for $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}$.

Indeed, the solution $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}$ is described by the following formulae.

Define on the complex plane, cut along the negative imaginary axis, the matrix function,

$$\Psi_0(\zeta) := \begin{pmatrix} 2^{\alpha}\zeta^{\alpha}\psi(\alpha+\beta,1+2\alpha,2e^{\frac{i\pi}{2}}\zeta)e^{i\pi\left(2\beta+\frac{3\alpha}{2}\right)}e^{-i\zeta} & -2^{\alpha}\zeta^{\alpha}\psi(1+\alpha-\beta,1+2\alpha,2e^{-\frac{i\pi}{2}}\zeta)e^{i\pi\left(\beta+\frac{3\alpha}{2}\right)}e^{i\zeta}\frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)} \\ -2^{-\alpha}\zeta^{-\alpha}\psi(1-\alpha+\beta,1-2\alpha,2e^{\frac{i\pi}{2}}\zeta)e^{i\pi\left(\beta-\frac{7\alpha}{2}\right)}e^{-i\zeta}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)} & 2^{-\alpha}\zeta^{-\alpha}\psi(-\alpha-\beta,1-2\alpha,2^{-\frac{i\pi}{2}}\zeta)e^{-\frac{3i\pi}{2}\alpha}e^{i\zeta} \end{pmatrix}, \quad \text{(A.27)}$$

where the branches of the multi-valued functions in the right hand side of the equation (including the confluent hypergeometric function) are fixed by the condition,

$$-\frac{\pi}{2} < \arg\zeta < \frac{3\pi}{2}.$$

We use I–VIII to denote the eight sectors as in Figure 11. The function $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$ is given

then by the equations,

$$\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta) = 2^{\beta\sigma_3} e^{\frac{i\pi}{2}\beta\sigma_3} \begin{pmatrix} e^{-i\pi(\alpha+2\beta)} & 0 \\ 0 & e^{i\pi(2\alpha+\beta)} \end{pmatrix}$$

$$\times \Psi_0(\zeta) \begin{cases} \begin{pmatrix} 0 & -e^{-\pi i\beta} \\ e^{\pi i\beta} & 0 \end{pmatrix} & \text{if } \zeta \in I, \\ \begin{pmatrix} e^{2\pi i\alpha} & -e^{-\pi i\beta} \\ e^{\pi i\beta} & 0 \end{pmatrix} & \text{if } \zeta \in II, \\ I & \text{if } \zeta \in III, \\ \begin{pmatrix} 1 & 0 \\ e^{\pi i(\beta-2\alpha)} & 1 \end{pmatrix} & \text{if } \zeta \in IV, \\ \begin{pmatrix} e^{\pi i\alpha} & 0 \\ e^{\pi i(\beta-\alpha)} & e^{-\pi i\alpha} \end{pmatrix} & \text{if } \zeta \in V, \\ \begin{pmatrix} e^{\pi i\alpha} & 0 \\ 2i\sin\pi(\beta-\alpha) & e^{-\pi i\alpha} \end{pmatrix} & \text{if } \zeta \in VI, \\ \begin{pmatrix} 0 & -e^{i\pi(\alpha+\beta)} \\ e^{-\pi i(\alpha+\beta)} & -2ie^{i\pi\beta}\sin\pi(\beta-\alpha) \end{pmatrix} & \text{if } \zeta \in VII, \\ \begin{pmatrix} e^{-i\pi\alpha} & -e^{i\pi(\alpha+\beta)} \\ e^{\pi i(\beta-3\alpha)} & -2ie^{i\pi\beta}\sin\pi(\beta-\alpha) \end{pmatrix} & \text{if } \zeta \in VIII. \end{cases} \tag{A.28}$$

It is a straightforward though a bit involved calculation to check that the function $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}$ defined by (A.27)-(A.28) does indeed satisfy the jump conditions indicated in Figure 11. By a direct calculation, one can also establish the following asymptotic behavior of $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$ as $\zeta \to \infty$.

$$\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta) = \left(I + \frac{m_{\tilde{\Phi}}}{\zeta} + O\left(\frac{1}{\zeta^2}\right)\right) \zeta^{-\beta\sigma_3} e^{-i\zeta\sigma_3}$$

$$\times \begin{cases} I & \text{if } 0 < \arg\zeta < \frac{\pi}{2}, \\ e^{i\pi\alpha\sigma_3} & \text{if } \frac{\pi}{2} < \arg\zeta < \pi, \\ \begin{pmatrix} 0 & -e^{i\pi(\alpha+\beta)} \\ e^{-i\pi(\alpha+\beta)} & 0 \end{pmatrix} & \text{if } \pi < \arg\zeta < \frac{3\pi}{2}, \\ \begin{pmatrix} 0 & -e^{-i\pi\beta} \\ e^{i\pi\beta} & 0 \end{pmatrix} & \text{if } -\frac{\pi}{2} < \arg\zeta < 0, \end{cases} \tag{A.29}$$

where

$$m_{\tilde{\Phi}} = \begin{pmatrix} \frac{i}{2}(\beta^2-\alpha^2) & -\frac{i}{2}\frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)} e^{4\pi i\alpha+i\pi\beta} \\ \frac{i}{2}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)} e^{-4\pi i\alpha-i\pi\beta} & \frac{i}{2}(\alpha^2-\beta^2) \end{pmatrix}. \tag{A.30}$$

The asymptotic formulae (A.29) indicate that the local parametrix $P^{(0)}_{\alpha,\beta}(z)$ takes again the form,

$$P^{(0)}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)\Phi^{(CHF)}_{\alpha,\beta}(tf(z))e^{tg(z)\sigma_3}, \tag{A.31}$$

where the change-of-variable function $f(z)$ is exactly the same as before, i.e., as in the case $b = 1$, while the holomorphic at $z = 0$ matrix valued function $E_{\alpha,\beta}(z)$ is defined by the formula which is slightly more complicated than the previous equations (3.30). Indeed, this time we have,

$$E_{\alpha,\beta}(z) = \begin{cases} P^{(\infty)}(z)(tf(z))^{\beta\sigma_3} e^{2it\sigma_3/3} e^{-i\pi\left(\alpha+\frac{\beta}{2}\right)\sigma_3}, & \operatorname{Im} z > 0, \\ P^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (tf(z))^{\beta\sigma_3} e^{2it\sigma_3/3} e^{-i\pi\left(\alpha+\frac{\beta}{2}\right)\sigma_3}, & \operatorname{Im} z < 0, \end{cases} \tag{A.32}$$

with $0 < \arg f(z) < 2\pi$, and, in particular,

$$E_{\alpha,\beta}(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 2\beta - 1 & 2\beta + 1 \end{pmatrix} (4t)^{\beta\sigma_3} e^{\frac{2it}{3} - i\pi\left(\frac{1}{4}+\alpha-\frac{\beta}{2}\right)\sigma_3}. \tag{A.33}$$

Having constructed the local parametrix at $z = 0$, the further arguments are identical to the ones we have used in the case $b = 1$. As a result, we arrive at the following representation for the solution $C_\alpha(z)$ to the "master" C -RH problem (cf. (3.38)).

$$\begin{aligned} C_\alpha(z) &= D_{\alpha,\beta}(z) P^{(0)}_{\alpha,\beta}(z) \\ &= D_{\alpha,\beta}(z) E_{\alpha,\beta}(z) \Phi^{(CHF)}_{\alpha,\beta}(tf(z)) e^{tg(z)\sigma_3}, \end{aligned} \tag{A.34}$$

where $D_{\alpha,\beta}(z) = I + O((-s)^{-3/2})$ as $s \to -\infty$. Hence, similar to the case $b = 1$, we have from (3.18) and (A.34) that

$$\begin{aligned} u(s) &= \frac{i}{\sqrt{-s}} \left[E_{\alpha,\beta}(0) \lim_{z\to 0} \left(z \left(\frac{d}{dz} \Phi^{(CHF)}_{\alpha,\beta}(tf(z)) \right) \left(\Phi^{(CHF)}_{\alpha,\beta}(tf(z)) \right)^{-1} \right) E^{-1}_{\alpha,\beta}(0) \right]_{12} + O(s^{-2}) \\ &= \frac{i}{\sqrt{-s}} \left[E_{\alpha,\beta}(0) \lim_{\zeta\to 0} \left(\zeta \left(\frac{d}{d\zeta} \Phi^{(CHF)}_{\alpha,\beta}(\zeta) \right) \left(\Phi^{(CHF)}_{\alpha,\beta}(\zeta) \right)^{-1} \right) E^{-1}_{\alpha,\beta}(0) \right]_{12} + O(s^{-2}) \\ &= \frac{i}{\sqrt{-s}} \left[E_{\alpha,\beta}(0) \lim_{\zeta\to 0} \left(\zeta \left(\frac{d}{d\zeta} \widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta) \right) \left(\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta) \right)^{-1} \right) E^{-1}_{\alpha,\beta}(0) \right]_{12} + O(s^{-2}), \end{aligned} \tag{A.35}$$

as $s \to -\infty$. The last equality follows from the fact that the matrix function $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$ differs from the matrix function $\Phi^{(CHF)}_{\alpha,\beta}(\zeta)$ only by a non-singular piecewise constant right matrix multiplier, see (A.25).

The function $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$ solves the Riemann-Hilbert problem whose jump conditions are depicted in Figure 11, the asymptotic behavior as $\zeta \to \infty$ is indicated in (A.26), and the branching singularity at $\zeta = 0$ is of type described in (1.9) and (1.10). Indeed, if $\alpha - \frac{1}{2} \notin \mathbb{N}_0$, then

$$\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta) = B(\zeta) \begin{pmatrix} \zeta^\alpha & 0 \\ 0 & \zeta^{-\alpha} \end{pmatrix} Q, \tag{A.36}$$

where B is analytic at $\zeta = 0$. If $\alpha \in \frac{1}{2} + \mathbb{N}_0$, then there exists a constant κ such that

$$\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta) = B(\zeta) \begin{pmatrix} \zeta^{\alpha} & \kappa\zeta^{\alpha}\log\zeta \\ 0 & \zeta^{-\alpha} \end{pmatrix} Q, \tag{A.37}$$

where B is again analytic at $\zeta = 0$. The right matrix multipliers Q in (A.36) and (A.37) are piecewise constant matrix functions; in fact, they are constant in the eight sectors I–VIII. The matrices Q can be written down explicitly using either the general algebraic properties of the Riemann-Hilbert problem, or the explicit formulae (A.27) - (A.28) for the function $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$. However, we won't do this. The important feature of the matrices Q as well as of all the jump matrices of the $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}$ - RH problem, is that they are constant with respect to ζ. Therefore, we can exploit the standard arguments of the theory of integrable systems (see e.g. [16], [8]; see also [10], Chapters 2, 3) and conclude that

$$\left(\frac{d}{d\zeta}\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)\right)\left(\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)\right)^{-1} \equiv -i\zeta\sigma_3 + \frac{A_{-1}}{\zeta}, \tag{A.38}$$

where

$$A_{-1} = -\beta\sigma_3 + i[\sigma_3, m_{\tilde{\Phi}}] = \begin{pmatrix} -\beta & \frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)}e^{4\pi i\alpha + i\pi\beta} \\ \frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)}e^{-4\pi i\alpha - i\pi\beta} & \beta \end{pmatrix}. \tag{A.39}$$

Indeed, the ζ-independence of all the jump matrices implies that the function[5],

$$A(\zeta) := \left(\frac{d}{d\zeta}\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)\right)\left(\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)\right)^{-1},$$

is analytic on $\mathbb{C} \setminus \{0\}$. Moreover, from the behavior of $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$ at $\zeta = 0$ (see (A.36), (A.37)) and at $\zeta = \infty$ (see (A.29)), we conclude that $A(\zeta)$ is in fact a rational function which has the only simple pole at $\zeta = 0$ and such that $A(\infty) = i\sigma_3$. Hence, by Liouville's theorem,

$$A(\zeta) = -i\sigma_3 + \frac{A_{-1}}{\zeta}.$$

This proves (A.38). Equation (A.39) is obtained by substituting the asymptotic expansion (A.29) into the left hand side of (A.38) and equating the terms of the order [6] $O(\zeta^{-1})$.

[5]All the jump matrices of the $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}$ - RH problem are having the unit determinant. Together with the asymptotic condition (A.26) this yield the identity, $\det\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(z) \equiv 1$ and therefore the holomorphic invertability of the matrix $\widetilde{\Phi}^{(CHF)}_{\alpha,\beta}(\zeta)$ for all ζ.

[6]Of course, equation (A.38) can be deduced via the direct differentiation of (A.27) and use of the classical confluent hypergeometric equation. This derivation though is much more involved then the one we just used.

Relation (A.38) allows us to re-write the asymptotic formula (A.35) as

$$u(s) = \frac{i}{\sqrt{-s}}\Big[E_{\alpha,\beta}(0)A_{-1}E^{-1}_{\alpha,\beta}(0)\Big]_{12} + O(s^{-2}), \quad s \to -\infty. \tag{A.40}$$

Plugging in here (A.33) and (A.39) yields the asymptotic equation,

$$u(s) = \frac{i}{\sqrt{-s}}\left[\beta - \frac{i}{2}\frac{\Gamma(1+\alpha-\beta)}{\Gamma(\alpha+\beta)}e^{i\theta(s)} - \frac{i}{2}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha-\beta)}e^{-i\theta(s)}\right] + O\left(\frac{1}{s^2}\right), \quad s \to -\infty, \tag{A.41}$$

where

$$\theta(s) = \frac{4t}{3} - 2i\beta\log 8t - \alpha\pi, \qquad t = (-s)^{3/2}.$$

Taking into account that we have assumed that $b > 0$ and hence that β is pure imaginary, we transform (A.41) into our final asymptotic representation for the (real-valued) *tronquée* solutions of the thirty fourth Painlevé equation as $s \to -\infty$,

$$u^{(tronq)}(s) = \frac{\beta_0}{\sqrt{-s}} + \frac{\sqrt{\alpha^2+\beta_0^2}}{\sqrt{-s}}\cos\left(\frac{4}{3}(-s)^{3/2} - 3\beta_0\log(-s) + \chi\right) + O(s^{-2}), \tag{A.42}$$

where

$$\beta_0 \equiv i\beta = -\frac{1}{2\pi}\log b,$$

and the phase χ is given explicitly in terms of β_0 and α,

$$\chi = -\alpha\pi - 6\beta_0\log 2 + \arg(\alpha + i\beta_0) + 2\arg\Gamma(\alpha + i\beta_0). \tag{A.43}$$

Remark A.1 The above asymptotic analysis of the Painlevé XXXIV Riemann-Hilbert problem does not need the condition $b > 0$ and hence the pure imaginary β. The necessary restriction is in fact,

$$|\mathrm{Re}\,\beta| < \frac{1}{2}.$$

With this restriction, we again arrive to the more general formula (A.41), but with the change of the error term

$$O\left(\frac{1}{s^2}\right)$$

to the term

$$O\left(\frac{1}{s^{2-3|\mathrm{Re}\,\beta|}}\right).$$

The asymptotic analysis of the general *tronquée* solution which we just performed makes a strong case in favor of the fact that the solution $u_\alpha(s)$ is characterized uniquely by the asymptotic conditions of Theorem 1.2. Indeed, using the transformation formula (A.3), we conclude that every *tronquée* solution of Painlevé XXXIV, i.e., one behaving at $s \to +\infty$ as (1.7), maps to a *tronquée* solution of Painlevé II. This means that the Painlevé II Stokes multipliers must be as in (A.15), because otherwise the asymptotics is oscillatory for $s \to -\infty$ and elliptic in the sectors $2\pi/3 < \arg s < \pi$ and $\pi < \arg s < 4\pi/3$ (see [17] and [12]). This in turns means that the asymptotic condition (1.7) selects the *tronquée* family (A.18) of solutions. As we have already shown, for the positive b, in fact for all b such that $|\arg b| < \pi$ it must be exactly $b = 1$ in order for the solution to behave at $-\infty$ as it is indicated in (1.8). It is natural to expect that in the case $b < 0$ the "minus infinity" asymptotics of the solution is different from (1.8).

As it was mentioned in the introduction, we actually conjecture that the asymptotics (1.8) alone fixes the solution uniquely. The supporting arguments are based on the fact that the substitution of the asymptotic ansatz (A.42) into the formula (A.3) yields the asymptotic representation for the Painlevé II function,

$$q(s) \sim \sqrt{\frac{s}{2}} \cot\left(\frac{\sqrt{2}}{3}s^{3/2} - \frac{3}{2}\beta_0 \log s + \hat{\chi}\right) \qquad \text{as } s \to \infty. \tag{A.44}$$

This asymptotics is consistent with the general results of [17] for real-valued solutions of Painlevé II which provide the explicit formulae relating the Painlevé II monodromy data a_1, a_2 and a_3 and the asymptotic parameters β_0 and $\hat{\chi}$. This, in turn, allows us to establish a one-to-one correspondence between the Painlevé XXXIV monodromy data b_1, b_2 and b_4 and the asymptotic parameters β_0 and χ. Hence the uniqueness of the real-valued solution of the Painlevé XXXIV equation (1.3) with the asymptotic condition (1.8).

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